National Library of Canada
Bibliothèque nationale du Canada

Canadian Theses Service
Service des thèses canadiennes

Ottawa, Canada
K1A 0N4

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

NL-339 (r. 88/04) c

Canada

University of Alberta

On Nilpotence and Dimension in Group Rings

by
David Michael Riley

A Thesis
Submitted to the Faculty of Graduate Studies and Research
in Partial Fulfillment of the Requirements for the Degree
of Doctor of Philosophy

Department of Mathematics

—

Edmonton, Alberta
Spring 1991

National Library of Canada
Bibliothèque nationale du Canada

Canadian Theses Service
Service des thèses canadiennes

Ottawa, Canada
K1A 0N4

The author has granted an irrevocable non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of his/her thesis by any means and in any form or format, making this thesis available to interested persons.

The author retains ownership of the copyright in his/her thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without his/her permission.

L'auteur a accordé une licence irrévocable et non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de sa thèse de quelque manière et sous quelque forme que ce soit pour mettre des exemplaires de cette thèse à la disposition des personnes intéressées.

L'auteur conserve la propriété du droit d'auteur qui protège sa thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

ISBN 0-315-66723-0

Canada

UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: David Michael Riley

TITLE OF THESIS:

On Nilpotence and Dimension in Group Rings

DEGREE FOR WHICH THESIS WAS PRESENTED: Doctor of Philosophy

YEAR THIS DEGREE GRANTED: 1991

Permission is hereby granted to THE UNIVERSITY OF ALBERTA LIBRARY to reproduce single copies of this thesis and to lend or sell such copies for private, scholarly or scientific research purposes only.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

(Signed) David Riley

Permanent Address:
Department of Mathematics
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

Date: April 25, 1991

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **On Nilpotence and Dimension in Group Rings** submitted by **David Michael Riley** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

S K Sehgal

Supervisor: S.K. Sehgal

Gerald Cliff

G.H. Cliff

R.J. Elliott

A. Pianzola

A. Weiss

A. Lichtman

External Examiner: A. Lichtman

Date: March 22, 1991

To Pratima

ABSTRACT

This thesis is an investigation of three primary structures associated to a group ring. Namely the group of multiplicative units, the augmentation ideal, and the associated graded group ring.

The first main result is that if K is a field of characteristic $p > 0$ and G is a group with a p–element, then the unit group of the group algebra KG is hypercentral if and only if it is nilpotent. To see this, an explicit group-theoretic characterization of this property is given.

The second major area of study deals with the identification of the restricted Lie dimension subgroups. While it is known that the Lie and restricted Lie powers of the augmentation ideal can be quite different in general, it is proven that the Lie and restricted Lie dimension subgroups coincide for group algebras over fields of characteristics $p \neq 2, 3$.

Next, the notions of dimension and Lie dimension subgroups are generalized. Using techniques similar to those established for ordinary dimension subgroups, these generalized dimension subgroups and some of their properties are identified for all group rings over fields. In the process a unified proof for the corresponding ordinary and Lie dimension results is given. Some applications are made to some major problems in the area, including the modular isomorphism problem and the dimension subgroup problem.

Finally, we investigate when $gr(KG)$, the associated graded group ring of a group algebra KG, satisfies a polynomial identity. To do so, we use the well known fact that $gr(KG)$ is the restricted universal enveloping algebra of a certain canonical restricted Lie algebra associated to the group G. The primary result is that if $gr(KG)$ satisfies some polynomial identity, then $gr(KG)$ is Lie n–Engel, for some positive integer n.

ACKNOWLEDGMENT

This thesis would not have been written had it not been for the support that I have received from many past and present members of the Department of Mathematics. The friendship and guidance offered to me by David McLaughlin, Sabri Öncü, Arturo Pianzola, René Poliquin, Akbar Rhemtulla, Jim Semple, Mazi Shirvani, Sivakumar and Jon Vanderwerff shall never be forgotten. But above all, I am deeply indebted to my teacher Professor Sudarshan Sehgal. He performed his rôle as my supervisor with dedication and enthusiasm, and he and his family have shown to me more kindness than I can ever repay.

TABLE OF CONTENTS

CHAPTER		PAGE
Introduction		1
1. Group rings with hypercentral unit groups		5
	1.1 Introduction	5
	1.2 The modular case	5
	1.3 The characteristic zero case	21
2. Restricted Lie dimension subgroups		25
	2.1 Introduction	25
	2.2 Characteristic $p \geq 5$	26
	2.3 Characteristic zero	29
	2.4 Residual Lie nilpotence	30
	2.5 The exceptional characteristics $p = 2, 3$	31
3. Generalized dimension subgroups		34
	3.1 Introduction	35
	3.2 Basic lemmas	36
	3.3 $E^K_{m,k}(G)$ when K is a field of positive characteristic	38
	3.4 $E^K_{m,k}(G)$ when K is a field of characteristic zero	43
	3.5 On the dimension subgroup conjecture	45
	3.6 k-Nilpotence of the augmentation ideal	47
	3.7 Residual k-nilpotence of the augmentation ideal	50
	3.8 On the modular isomorphism problem	53
	3.9 The graded Lie algebra $\mathcal{L}^K_k(G)$ associated with $\{F_{m,k}(G)\}_{m\geq 1}$	55
4. Polynomial identities in graded group rings		59
	4.1 Introduction	59
	4.2 Lie nilpotent enveloping algebras	64
	4.3 Lie nilpotent graded enveloping algebras	67
	4.4 Lie n–Engel graded enveloping algebras	69
	4.5 Graded group rings	74
References		78

INTRODUCTION

This thesis is an investigation into three main aspects of group rings. Of primary emphasis are group rings over fields. One way of obtaining insight into the structure of a group ring is to study its group of multiplicative units since it can be considered as a link between the defining group and the group ring itself. This will be our first area of study. Another way of gaining understanding about a group ring is to examine the properties of its two-sided ideals. Of paramount importance here is the augmentation ideal. Like all two-sided ideals, the powers of the augmentation ideal determine normal subgroups, which are known as the dimension subgroups. The identification of these subgroups is a very long standing problem. This problem, and its generalizations, will be our second area of study. Finally, attached to any group ring is its associated graded group ring constructed from the powers of the augmentation ideal. Its Lie structure will be our last topic of investigation.

The problem of classifying group rings with unit groups having such properties as solubility or nilpotency has been studied by many authors (see [S] for an overview). Chapter One examines some group rings whose unit groups are hypercentral, that is, the ascending central series attains the group after some, possibly infinite, ordinal. This property is clearly a generalization of nilpotence. In fact a hypercentral group is always locally nilpotent; however, it need not even be soluble. Despite this fact, our main result is that if K is a field of characteristic $p > 0$ and G is a group containing a p-element, then the unit group of KG is hypercentral if and only if it is nilpotent. To see this, an explicit group-theoretic characterization of this property is given. This characterization was first given by Khripta [K]

for such modular group algebras with nilpotent unit groups. Group algebras over fields of characteristic zero with hypercentral unit groups are also classified, with a substantially different result.

Chapter Two deals with the identification of the so-called restricted Lie dimension subgroups. We can consider a group ring, or any other associative ring, as a Lie ring via the Lie bracket $[x, y] = xy - yx$. We would now like to define the Lie powers of the augmentation ideal to be the terms of its Lie lower central series. However, we require associative ideals. If we take then the associative ideals generated by the terms of the Lie lower central series, we obtain what are known as the restricted Lie powers of the augmentation ideal. The normal subgroups that they determine are the restricted Lie dimension subgroups. But there is another natural way of defining Lie powers of the augmentation ideal (or any associative ring for that matter), that is to expand to associative ideals at each step of the series. These Lie powers yield the Lie dimension subgroups. While it is known that the Lie and restricted Lie powers can be quite different in general, we prove that the Lie and restricted Lie dimension subgroups coincide for group algebras over fields of characteristics $p \neq 2, 3$. The proof uses the classification of Lie dimension subgroups given by Passi and Sehgal in [PS2], and a new result on the restricted Lie powers of the augmentation ideal given by Sharma and Srivastava [SS].

In Chapter Three we generalize the notions of dimension and Lie dimension subgroups. Using Sandling's formula [Sa1] for the Lie powers of the augmentation ideal as a starting point, we define for each $k \geq 0$ a k-dimension theory, where $k = 0$ and $k = 1$ correspond, respectively, to ordinary and Lie dimension theory. Using techniques similar to those used by Lazard for ordinary dimension subgroups in his fundamental paper [L] and those of Passi and Sehgal in their Lie dimension version [PS2], we identify our new k-dimension subgroups for all group rings over fields of

positive characteristic. In the process we give a unified proof for the classification of dimension and Lie dimension subgroups over fields given in [L] and [PS2]. These new subgroups turn out to be the same as those subgroups at the heart of a series of new works by Shalev about modular group algebras (see References). Our alternate description will enable us to broaden his results. This takes us to the end of section 3.3.

Subsequently in section 3.4, we identify these k-dimension subgroups over fields of characteristic zero, and demonstrate that Shalev's results do not completely transfer to this situation. After that, in section 3.5, we have a short look at our k-dimension subgroups over the integers and their connection with the dimension subgroup conjecture.

Sections 3.6 and 3.7 include comments that the some of the major results on the nilpotence and Lie nilpotence of the augmentation ideal can be extended to encompass our new k-dimension theory.

As an application to some of our aforementioned results, the modular isomorphism is examined in section 3.8, where some new (abelian) sections of a group G are shown to be determined by the $\mathbf{F}_p$–algebra isomorphism class of $\mathbf{F}_pG$.

Finally, in section 3.9 we investigate the canonical restricted Lie algebra associated with our series of k-dimension subgroups. In particular, we generalize Quillen's well known result [Q] about the graded ring of a group algebra being the restricted universal enveloping algebra of the canonical Lie algebra associated with the series of ordinary dimension subgroups.

In our last chapter, Chapter Four, we study restricted Lie algebras, L. We wish to discover when the restricted universal enveloping algebra, $u(L)$, of L satisfies different classes of polynomial identities. Passman has already given a general answer to the problem of describing when $u(L)$ satisfies some polynomial identity

in [P2]. Our preliminary results enable us to simplify Passman's theorem, which we subsequently exploit to our advantage. In section 4.2, it is determined precisely when $u(L)$ is strongly Lie nilpotent, which is to say its Lie powers eventally become zero. Then, in sections 4.3 and 4.4, with the goal in mind of applying our results to the associated graded group ring of a group algebra, we specialize our class of restricted Lie algebras to just those that admit an $\mathbb{N}$–grading. In what turns out to be our main argument, we prove that, in this case, the fact that $u(L)$ satisfies some polynomial identity implies that $u(L)$ satisfies a polynomial identity of the form $[x, y, y, \ldots, y] = 0$, which is to say $u(L)$ is Lie n–Engel, for some n. We conclude the chapter with section 4.5, in which we apply our results to graded group rings, for, as mentioned previously, Quillen has shown in [Q] that a graded group ring is a restricted universal enveloping algebra of a certain canonical $\mathbb{N}$–graded restricted Lie algebra associated to the defining group.

Chapter One

GROUP RINGS WITH HYPERCENTRAL UNIT GROUPS

1.1 Introduction.

A group G is said to be *hypercentral* if the ascending central series reaches G after some, possibly infinite, ordinal. Černikov [R1; 2.19] showed this to be equivalent to the property that for every sequence $x_1, x_2, \ldots$ of elements in G, there exists an index m such that the group commutator $(x_1, x_2, \ldots, x_m) = 1$. So, in particular, if m can be chosen independently of the sequence, then G is nilpotent.

The problem of classifying group rings with unit groups having such properties as solubility or nilpotency has been studied by many authors (see [S] for an overview). Examined here are some group rings whose unit groups are hypercentral.

Let K be a field of characteristic $p \geq 0$. Denote by $\mathcal{U}(KG)$ the group of units of the group algebra KG. Inspired by the paper of Bovdi and Khripta [BK1] in which group algebras that are Lie hypercentral are classified, the goal of this chapter is to describe precisely the structure of KG when $\mathcal{U}(KG)$ is hypercentral in case that $p = 0$ or $p > 0$ and G contains a $p-$element. It must be noted that together Bovdi and Khripta [BK2] have independently obtained the following theorem, which is the primary result of this chapter.

1.2 The modular case.

1.2.1 THEOREM. *Let p be any prime and let G be any group containing p-elements. If K is a field of characteristic p, then the following are equivalent:*

(i) $\mathcal{U}(KG)$ is hypercentral;

(ii) $\mathcal{U}(KG)$ is nilpotent; and

(iii) G is nilpotent and the derived subgroup G' of G is a finite p-group.

Since (ii) $\Rightarrow$ (i) is trivial and (ii) $\Leftrightarrow$ (iii) is a result of Khripta [K and S; pp. 179-181], it is enough to show only that (i) $\Rightarrow$ (iii). However, for the sake of completeness, we include an outline of a proof of (iii) $\Rightarrow$ (i).

In addition to conventional notation, the following will be used: $(x,y) = xyx^{-1}y^{-1}$ is the group commutator of group elements x and y; $\phi(G)$ denotes the FC-centre of the group G; if h is a group element of finite order, let $\hat{h} = \sum_{i=1}^{o(h)} h^i$; Dr denotes a restricted direct product; and if $N \triangleleft G$, then $\Delta(G,N)$ is the kernel of the natural epimorphism $KG \to K(G/N)$ induced by $G \to G/N$. When $N = G$, $\Delta(G,G)$ is the augmentation ideal of KG, which is abbreviated by just $\Delta(G)$.

Proof of 1.2.1 (iii) $\Rightarrow$ (i). First we make some definitions and prove some results that will not be needed again in this chapter. Define the Lie powers of $\Delta(G)$ inductively by $\Delta^{(1)}(G) = \Delta(G)$ and $\Delta^{(m+1)}(G) = [\Delta^{(m)}(G), \Delta(G)]KG$, the ideal of KG generated by the Lie products $[x,y] = xy - yx$, where $x \in \Delta^{(m)}(G)$ and $y \in \Delta(G)$. Let $\epsilon(x)$ be the augmentation of $x \in KG$.

We claim now that $\mathcal{U}_m(KG) - 1 \subseteq \Delta^{(m)}(G)$ for all $m \geq 2$, where $\mathcal{U}_m(KG)$ is the m-th term of the lower central series of $\mathcal{U}(KG)$. Indeed, if $u, v \in \mathcal{U}(KG)$, then

$$(u,v)-1 = (uv-vu)u^{-1}v^{-1} = [u,v]u^{-1}v^{-1} = [u-\epsilon(u), v-\epsilon(v)]u^{-1}v^{-1} \subseteq \Delta^{(2)}(G).$$

Now using the identity $(ab-1) = (a-1)+(b-1)+(a-1)(b-1)$, we see that the claim is true for $m = 2$, and then by induction for all $m \geq 2$.

Thus to show $\mathcal{U}(KG)$ is nilpotent when G is nilpotent and G' is a finite p-group, it suffices to prove that in this case $\Delta^{(m)}(G) = 0$ for some m. To see this, we proceed by induction on $|G'|$.

We can find an element z of order p in $G' \cap \zeta(G)$, where $\zeta(G)$ is the centre of G. Observe that $\overline{G} = G/\zeta(G)$ is a nilpotent group with $|\overline{G}'| < |G'|$. Consequently, by the induction hypothesis, there is an m with $\Delta^{(m)}(\overline{G}) = 0$, in other words,

$$\Delta^{(m)}(G) \subseteq \Delta(G, \langle z \rangle) = (z-1)KG.$$

It follows by the centrality of z and a simple induction argument that

$$\Delta^{(mp)}(G) \subseteq (z-1)^p KG = 0.$$

This finishes the proof.

□

Unless specified to the contrary, *throughout the remainder of this section, K is assumed to be a field of characteristic $p > 0$ and G is a group containing a p-element such that $\mathcal{U}(KG)$ is hypercentral.* We now devote our attention to proving the implication (i) ⇒ (iii).

1.2.2 LEMMA. *If G is any hypercentral group containing a p-element, then $\zeta(G)$, the centre of G, contains an element of order p.*

Proof. By [R2; 12.2.4], G hypercentral implies that G is locally nilpotent. Let P be the set of p-elements in G. Then $1 \neq P \lhd G$ by [R2; 12.1.1], so that since G is hypercentral, $P \cap \zeta(G) \neq 1$ by [R1; 2.16].

□

The following is a generalization of an unpublished result of Jairo Gonçalves.

1.2.3 PROPOSITION. *G is an FC-group.*

Proof. Since $\mathcal{U}(KG)$ being hypercentral implies that G is hypercentral, by Lemma 1.2.2 we may pick $h \in \zeta(G)$ such that $o(h) = p$. Set $\eta = \hat{h}$. Then $\eta^2 = 0$

so that $1 + \eta\alpha \in \mathcal{U}(KG)$ with inverse $1 - \eta\alpha$, for any $\alpha \in KG$. Assume to the contrary that G is not an FC-group. Since G is hypercentral, the ascending central series of G reaches G. Thus let λ be the first ordinal such that the λth centre of G, $\zeta_\lambda(G) \not\subseteq \phi(G)$. Notice that λ cannot be a limit ordinal since if $\zeta_\alpha(G) \subseteq \phi(G)$ for all $\alpha < \lambda$, then $\zeta_\lambda(G) = \bigcup_{\alpha<\lambda} \zeta_\alpha(G) \subseteq \phi(G)$ as well. Hence, in particular, $(G, \zeta_\lambda(G)) \leq \zeta_{\lambda-1}(G)$. Thus by Poincaré's theorem [R2; 1.3.12], if $c_1, c_2, \ldots, c_r \in \zeta_{\lambda-1}(G) \subseteq \phi(G)$, then $[G : C_G\{c_1, \ldots, c_r\}] < \infty$. Now pick any $g \in \zeta_\lambda(G) \backslash \phi(G)$.

We construct a sequence (g_n) with the following properties:

(1) $g_{n+1} \in C_G\{b_1, \ldots, b_n\}$ where $b_j = (g_j, g) \in \zeta_{\lambda-1}(G)$; and

(2) $\eta(1 - b_1)(1 - b_2)\ldots(1 - b_{n+1}) \neq 0$ for all n.

Since $[G : C_G(g)] = \infty$, we can find $g_1 \in G$ such that $\eta(1 - b_1) \neq 0$ and $b_1 = (g_1, g) \in \zeta_{\lambda-1}(G)$. For recall that the right annihilator of a nonzero element of KG contains only finitely many elements of the form $g - 1$, $g \in G$. Suppose now we have selected $g_1, g_2, \ldots, g_n$ in G satisfying (1) and (2). Let $H = C_G\{b_1, \ldots, b_n\}$. We claim $[H : C_H(g)] = \infty$. Indeed, if $[H : C_H(g)] < \infty$, it follows that $[G : C_G(g)] \leq [G : C_H(g)] = [G : H][H : C_H(g)] < \infty$, contradicting our choice of g. Hence $[H : C_H(g)] = \infty$, and we can choose an element $g_{n+1} \in H$ such that $b_{n+1} = (g_{n+1}, g) \in \zeta_{\lambda-1}(G)$ and $\eta(1 - b_1)(1 - b_2)\ldots(1 - b_{n+1}) \neq 0$. Thus the sequence may be constructed.

Finally, we claim $(1 + \eta g, g_1, g_2, \ldots, g_n) = 1 + \eta(1 - b_1)(1 - b_2)\ldots(1 - b_n)g$, for all $n \geq 1$, contradicting that $\mathcal{U}(KG)$ is hypercentral. Indeed,

$$(1 + \eta g, g_1) = (1 + \eta g)g_1(1 - \eta g)g_1^{-1}$$

$$= 1 + \eta g - \eta g_1 g g_1^{-1} = 1 + \eta(1 - b_1)g,$$

so by induction we may assume the result for $n \geq 1$. Then

$$(1 + \eta g, g_1, \ldots, g_{n+1}) = (1 + \eta(1 - b_1) \ldots (1 - b_n)g, g_{n+1})$$

$$= 1 + \eta(1 - b_1) \ldots (1 - b_n)g - \eta(1 - b_1) \ldots (1 - b_n)g_{n+1}gg_{n+1}^{-1}$$

$$= 1 + \eta(1 - b_1) \ldots (1 - b_{n+1})g.$$

This proves the proposition.

□

1.2.4 LEMMA. *For every $x, y \in G$, there is an $m \in \mathbf{N}$ (depending on x and y) such that $(x, y^{p^m}) = 1$.*

Proof. [S; VI. 3.2]

1.2.5 LEMMA. *G' is a p-group.*

Proof. We may assume G is finitely generated. Then G is a finitely generated FC-group by Proposition 1.2.3, so that G is residually finite [S; I.4.7]. Since G' is a torsion group [R2; 14.5.9], let $g \in G'$ have prime order q. We can find $N \triangleleft G$ such that G/N is finite and $g \notin N$. Since G/N is finite and satisfies the hypotheses of Lemma 1.2.4, there is an integer m such that $(G/N)^{p^m} \subseteq \zeta(G/N)$. So by Schur's lemma [S; I.4.2], then $((G/N)')^{p^t} = 1$ for some integer t, so that $(G/N)' = G'N/N$ is a p-group. Hence $p = q$ since $1 \neq gN \in G'N/N$, proving the lemma.

□

1.2.6 PROPOSITION. *$G' \cap \zeta(G)$ cannot contain a direct product of infinitely many nontrivial subgroups.*

Proof. Suppose $\underset{i \in \mathbf{N}}{Dr} H_i \subseteq G' \cap \zeta(G)$ with $1 \neq H_i$ for all i. Then, in particular, the p-group $G' \cap \zeta(G)$ contains a subgroup $H = \underset{i \in \mathbf{N}}{Dr} C_i$, where each C_i is a cyclic group of order p.

We construct sequences (g_n) and (h_{2n-1}) in G with the following properties:

(1) $g_1, g_2 \in G$ are such that $(g_2, g_1) = b_1 \neq 1$;

(2) $h_1 \in H \backslash \langle b_1 \rangle$;

(3) $g_{2n+1}, g_{2n+2} \in C_G\{g_1, g_2, \ldots, g_{2n}\}$ are such that $(g_{2n+2}, g_{2n+1}) = b_{2n+1} \notin \langle b_1, b_3, \ldots, b_{2n-1}, h_1, h_3, \ldots, h_{2n-1} \rangle$; and

(4) $h_{2n+1} \in H \backslash \langle b_1, b_3, \ldots, b_{2n+1}, h_1, h_3, \ldots, h_{2n-1} \rangle$ for all n.

Certainly such g_1, g_2 and h_1 can be found. Suppose as well that for $n \geq 1$, $g_1, g_2, \ldots, g_{2n}$ and $h_1, h_3, \ldots, h_{2n-1}$ satisfy (1)–(4). Since G is an FC-group by Proposition 1.2.3, $[G : C] < \infty$ where $C = C_G\{g_1, g_2, \ldots, g_{2n}\}$. Hence by [T; 7.5], C' is infinite since G' is. Since G' is locally finite, so is C'. Thus C' contains infinitely many commutators $b_{2n+1} = (g_{2n+2}, g_{2n+1})$ with $g_{2n+1}, g_{2n+2} \in C$. Since $F_1 := \langle b_1, b_3, \ldots, b_{2n-1}, h_1, h_3, \ldots, h_{2n-1} \rangle \subseteq G'$ is finite, we may pick $b_{2n+1} \notin F_1$. Similarly since $F_2 := \langle b_1, b_3, \ldots, b_{2n+1}, h_1, h_3, \ldots, h_{2n-1} \rangle$ is finite and H is not, we can choose $h_{2n+1} \in H \backslash F_2$. Thus the sequences can be constructed.

Observe that since $o(h_{2n+1}) = p$, clearly (4) implies that for all $n \geq 0$,

$$\langle h_{2n+1} \rangle \cap \langle b_1, b_3, \ldots, b_{2n+1}, h_1, h_3, \ldots, h_{2n-1} \rangle = 1$$

This together with the fact that $b_{2n-1} \notin \langle b_1, b_3, \ldots, b_{2n-3}, h_1, h_3, \ldots, h_{2n-3} \rangle$ for all $n \geq 1$ implies easily that

$$\hat{h}_1 \hat{h}_3 \ldots \hat{h}_{2n+1}(1 - b_1)(1 - b_3) \ldots (1 - b_{2n-1}) \neq 0$$

for all $n \geq 1$ since the coefficient of 1 is exactly 1.

Finally, by an easy induction (see [S; p. 181] and [BK1; Lemma 5]), we have

for all $n \geq 1$

$$(1 + \hat{h}_1 g_1,\ 1 + \hat{h}_3 g_2 g_3,\ 1 + \hat{h}_5 g_4 g_5, \ldots, 1 + \hat{h}_{2n+1} g_{2n} g_{2n+1})$$

$$= 1 + \hat{h}_1 \hat{h}_3 \ldots \hat{h}_{2n+1}(g_1, g_2 g_3, \ldots, g_{2n} g_{2n+1})$$

$$= 1 + \hat{h}_1 \hat{h}_3 \ldots \hat{h}_{2n+1}(1 - b_1)(1 - b_3) \ldots (1 - b_{2n-1}) g_1 g_2 \cdots g_{2n+1}$$

$$\neq 1.$$

This contradicts the fact that $\mathcal{U}(KG)$ is hypercentral, and the proposition is proved.

□

1.2.7 DEFINITION. *If G is a possibly nonabelian group, let us say that a p-element g_0 in G has infinite p-height if we can find a g_1 in G such that $g_1^p = g_0$ and recursively a g_{i+1} in G satisfying $g_{i+1}^p = g_i$ for each $i \geq 1$.*

Notice that if $j \geq i$ then $g_j^{p^{j-i}} = g_i$, so that g_i and g_j commute. Also observe $o(g_i) = p^i o(g_0)$. Thus since $o(g_0)$ is assumed to be a power of p, for each $i \geq 1$, g_i also has infinite p-height. It now becomes apparent that the union of the subgroups

$$\langle g_0 \rangle \leq \langle g_1 \rangle \leq \langle g_2 \rangle \leq \cdots$$

is isomorphic to the Prüfer p-group $Z(p^\infty)$ (see [R2; p. 23]). The converse is easily seen to be true as well: if G contains a subgroup isomorphic to $Z(p^\infty)$, then G contains a nontrivial element of infinite p-height.

1.2.8 LEMMA. *For any FC-group G, let $Q(G) = \{g \in G : g \text{ has infinite } p\text{-height}\}$. Then we have the following:*

(i) $Q(G)$ is a central subgroup of G;

(ii) $Q(G/Q(G)) = 1$;

(iii) if F is a central subgroup of G of finite exponent a power of p, then $Q(G) = 1$ implies $Q(G/F) = 1$; and

(iv) if we further assume G is nilpotent and F is a finite normal p-subgroup of G, then $Q(G) = 1$ implies $Q(G/F) = 1$.

Proof. (i) Suppose G contains an element g_0 of infinite p-height. Then for each $i \geq 1$, we can choose g_i inductively such that $g_i^p = g_{i-1}$. If $g_0 \notin \zeta(G)$, then we can find an $x \notin C_G(g_0)$. Then since $g_i^{p^i} = g_0$, certainly $x \notin C_G(g_i)$ for all $i \geq 0$. By assumption g_0 is a p-element, so let $o(g_0) = p^r$. Since G is FC, the set $\{x^{g_i} : i \geq 0\}$ is finite. Thus we can produce integers $i,\ j \geq 0$ such that $j > i + r$ and satisfying $x^{g_i} = x^{g_j}$, that is, $x \in C_G(g_i^{-1}g_j)$. Notice that since $o(g_0) = p^r$, then $o(g_i) = p^r p^i = p^{i+r}$. Hence we get $(g_i^{-1}g_j)^{p^{i+r}} = g_j^{p^{i+r}} = g_{j-(i+r)}$, so that $x \in C_G(g_{j-(i+r)})$, a contradiction. This shows g_0 was in fact central.

It remains to show $Q(G)$ is a subgroup of G. Indeed, if g_0 and h_0 both have infinite p-height, there exists $\{g_i\}$ and $\{h_i\}$ of G such that $g_i^p = g_{i-1}$ and $h_i^p = h_{i-1}$ for each $i \geq 1$. Since each g_i and h_i is of infinite p-height, they lie in $Q(G)$ and so are central as shown above. Hence $(g_i h_i)^p = g_i^p h_i^p = g_{i-1}h_{i-1}$ for each $i \geq 1$. Also since g_0 and h_0 are commuting p-elements, $g_0 h_0$ too is a p-element. We now see that $g_0 h_0$ has infinite p-height.

(ii) Suppose $g_0 Q(G) \in G/Q(G)$ has infinite p-height. Then there exists h_1 in G such that $(h_1 Q(G))^p = g_0 Q(G)$. Thus $g_0 = h_1^p a_0$ for some a_0 in $Q(G)$. Since $a_0 \in Q(G)$, we can find an a_1 in $Q(G)$ such that $a_1^p = a_0$. Then since $Q(G)$ is central, $g_0 = h_1^p a_0 = h_1^p a_1^p = (h_1 a_1)^p$. Let $g_1 = h_1 a_1$. Then $g_0 = g_1^p$. Now since $g_1 Q(G) = h_1 a_1 Q(G) = h_1 Q(G)$ has infinite p-height in $G/Q(G)$, we can similarly find an h_2 in G and b_1 in $Q(G)$ such that $g_1 = h_2^p b_1$. Since $b_1 \in Q(G)$, we can find $b_2 \in Q(G)$ such that $b_2^p = b_1$. Hence $g_1 = (h_2 b_2)^p$. Let $g_2 = h_2 b_2$. Then $g_1 = g_2^p$.

Proceeding in this manner shows that we can find recursively a g_i in G such that $g_i^p = g_{i-1}$ for each $i \geq 1$. Finally notice that since $g_0Q(G)$ is a p-element and elements of $Q(G)$ are p-elements, necessarily g_0 is a p-element. Thus $g_0 \in Q(G)$, and so $g_0Q(G) = 1$.

(iii) Suppose $1 \neq g_0F$ has infinite p-height. Then we can find inductively $g_i \in G$ and $f_i \in F$ such that $g_i^p = g_{i-1}f_i$ for $i \geq 1$. Pick j large enough that g_jF has order exceeding p^n, the exponent of F. Set $h_i = g_{j+i}^{p^n}$ for $i \geq 0$. Then $h_i^p = g_{j+i}^{p^{n+1}} = \left(g_{j+i-1}f_{j+i}\right)^{p^n} = g_{j+i-1}^{p^n} = h_{i-1}$ for $i \geq 1$ since F is central. Since g_0F is a p-element and F has exponent p^n, then g_0 and hence $h_0 = g_j^{p^n} \neq 1$ is a p-element. Thus we have shown $1 \neq h_0 \in Q(G)$.

(iv) We use induction on $|F|$. If $|F| = 1$, the result is trivial. So suppose now that the result holds for all nilpotent groups G_1 and finite normal p-subgroups F_1 of G_1 such that $|F_1| < |F|$. Since G is nilpotent and $F \neq 1$, $F \cap \zeta G \neq 1$. Hence if $F_1 = \frac{F}{F \cap \zeta(G)}$, then $|F_1| < |F|$. Now if $G_1 = \frac{G}{F \cap \zeta(G)}$, then $Q(G_1) = 1$ by (iii), so that by the induction hypothesis, $Q(G_1/F_1) = 1$. But clearly $G_1/F_1 \cong G/F$.

□

1.2.9 LEMMA. *(i) If A is any central p-subgroup of a group G, then $\Delta(G, A)$ is a nil ideal of KG.*

(ii) If I is any nil ideal of KG, then the natural projection $KG \to KG/I$ induces an epimorphism $\mathcal{U}(KG) \to \mathcal{U}(KG/I)$.

Proof. (i) By [S; p. 2], we know $\Delta(G, A) = \Delta(A)KG$, and any $\delta \in \Delta(G, A)$ can be written as $\delta = \sum_{i=1}^{n}(a_i - 1)\alpha_i$ where $a_i \in A$ and $\alpha_i \in KG$. Let $p^m = \max_i\{o(a_i)\}$. Then since A is central, we have by Frobenius' theorem $\delta^{p^m} = \sum_{i=1}^{n}(a_i^{p^m} - 1)\alpha_i^{p^m} = 0$.

(ii) The proof of [S; VI.3.3] uses only the fact that I is nil.

□

1.2.10 PROPOSITION. *Under our assumptions, if G' contains no subgroups isomorphic to the quasicyclic p-group, $Z(p^\infty)$, that is $Q(G') = 1$, then G is nilpotent and G' is a finite p-group.*

Proof. Let us show first that G is nilpotent. Thus we can assume $G' \neq 1$. By [T; 1.16] since G is FC and hypercentral, $\gamma_\omega(G) \subseteq \zeta(G)$. If $\gamma_\omega(G)$ is infinite, then $G' \cap \zeta(G)$ contains a countably infinite abelian p-group C without nontrivial elements of infinite p-height. By [R2; 4.3.15] then C is the direct product of (infinitely many) cyclic groups. However, this cannot be by Proposition 1.2.6, so that necessarily $\gamma_\omega(G)$ is finite.

Now since $\gamma_\omega(G)$ is a finite central p-subgroup of G, then $Q((G/\gamma_\omega(G))')$ $= Q(G'/\gamma_\omega(G)) = 1$ by Lemma 1.2.8(iii). Also by Lemma 1.2.9, $\Delta(G, \gamma_\omega(G))$ is nilpotent and hence $\mathcal{U}(KG/\Delta(G, \gamma_\omega(G))) \cong \mathcal{U}(K(G/\gamma_\omega(G)))$ is hypercentral. Clearly $G/\gamma_\omega(G)$ has p-elements as G' is a p-group by Lemma 1.2.5. Finally since $\gamma_\omega(G) \subseteq \zeta(G)$, and so $G/\gamma_\omega(G)$ is nilpotent if and only if G is nilpotent, we need only examine the case when $\gamma_\omega(G) = 1$.

If G is not nilpotent, then $\gamma_n(G) \neq 1$ for every $n \geq 1$. Thus we can find $1 \neq c_n \in \zeta(G) \cap \gamma_n(G)$ (since G is hypercentral and $1 \neq \gamma_n(G) \lhd G$) for every n. Then $\{c_n : n \geq 1\}$ is infinite for otherwise there exists an $i \geq 1$ such that $c_i \in \gamma_n(G)$ for infinitely many n, and hence $1 \neq c_i \in \gamma_\omega(G)$. But then $C = \langle c_n : n \geq 1 \rangle$ is a countably infinite abelian subgroup of $G' \cap \zeta(G)$ without elements of infinite p-height. This contradicts Proposition 1.2.6 as above; hence, G is indeed nilpotent.

It remains only to show that G' is finite. Suppose to the contrary that G' is infinite. Since G is nilpotent there then exists an integer r such that $\gamma_r(G)$ is infinite but $\gamma_{r+1}(G)$ is finite. Therefore since $\Delta(G, \gamma_{r+1}(G))$ is nilpotent by [S; I.2.21], then by Lemma 1.2.9(ii) $\mathcal{U}(K(G/\gamma_{r+1}(G)))$ is hypercentral and $G/\gamma_{r+1}(G)$ has p-elements. Also $Q(G') = 1$ implies $Q(G'/\gamma_{r+1}(G)) = 1$ by Lemma 1.2.8(iv).

Thus since $(G/\gamma_{r+1}(G))' = G'/\gamma_{r+1}(G)$ is finite if and only if G' is finite, we may assume $\gamma_{r+1}(G) = 1$. Hence $\gamma_r(G)$ is an infinite central subgroup of G'. As above, this contradicts Proposition 1.2.6, so that indeed G' is finite.

□

1.2.11 COROLLARY. *(i)* $(G/Q(G))' \cong G'/G' \cap Q(G)$ *is a finite p-group and* $G/Q(G)$ *is nilpotent.*

(ii) G'' *is a finite p-group and* G *is nilpotent.*

Proof. (i) Since $(G/Q(G))' = G'Q(G)/Q(G) \cong G'/G' \cap Q(G)$ is a p-group by Lemma 1.2.5, either $G/Q(G)$ has p-elements or it is abelian. So suppose without loss that $G/Q(G)$ has p–elements. By Lemma 1.2.8(ii), $Q(G/Q(G)) = 1$, so that in particular $Q((G/Q(G))') = 1$. By Lemmas 1.2.8(i) and 1.2.9 we also have that $\mathcal{U}(KG/\Delta(G, Q(G))) \cong \mathcal{U}(K(G/Q(G)))$ is hypercentral. Hence by Proposition 1.2.10, we are done.

(ii) Since $G' \cap Q(G) \subseteq G' \cap \zeta(G) \subseteq \zeta(G')$, by (i), $G'/\zeta(G')$ is finite. Hence by Schur's lemma, G'' is finite, too. Finally, since $Q(G) \subseteq \zeta(G)$ by 1.2.8(i), $G/Q(G)$ being nilpotent implies that G itself is nilpotent.

□

1.2.12 LEMMA. *If* x *is an element of order* $p^n > 1$ *in a group* G, *then* $(1-x)^{p^n-1} = \hat{x}$ *in* KG.

Proof. $(1-x)^{p^n-1} = \sum_{k=0}^{p^n-1} \binom{p^n-1}{k}(-1)^k x^k$ by the binomial theorem. But

$$\binom{p^n-1}{k} = \frac{(p^n-1)!}{k!(p^n-1-k)!}$$

$$= \frac{(p^n-1)(p^n-2)\ldots(p^n-k)}{k!} \equiv \frac{(-1)^k k!}{k!}$$

$$= (-1)^k \qquad \text{modulo } p.$$

Hence $(1-x)^{p^n-1} = \sum_{k=0}^{p^n-1} x^k = \hat{x}$ as claimed.

□

1.2.13 PROPOSITION. *G' is finite.*

Proof. Suppose to the contrary that G' is infinite. We first show that it is sufficient to consider only the case when $G' \cong Z(p^\infty)$.

Since G'' is finite by the preceding Corollary 1.2.11, we may assume G' is abelian by factoring by G''. Indeed $(G/G'')' = G'/G''$ is finite if and only if G' is finite. Also as G'' is a finite p-group, $\Delta(G, G'')$ is nilpotent by [S; I.2.21]; hence by Lemma 1.2.9(ii), $\mathcal{U}(K(G/G''))$ is hypercentral. Clearly G/G'' has p-elements as G' is a p-group.

Now by the structure theorem for divisible abelian groups, $G' \cap Q(G)$ is a direct product of copies of $Z(p^\infty)$ [R2; 4.1.5]. There can be only finitely many such factors by Lemma 1.2.8(i) and Proposition 1.2.6. Thus since $G'/G' \cap Q(G)$ is finite by the preceding corollary, G' is an abelian Černikov group, and hence by [R2; 5.4.23 and 4.2.11] we have $G' = \left(\underset{i=1}{\overset{t}{Dr}} Q_i\right) \times H$ where each $Q_i \cong Z(p^\infty)$ and H is finite. Since G' was assumed to be infinite, necessarily $t \geq 1$.

We now make the following reductions. Let $A = \underset{i=2}{\overset{t}{Dr}} Q_i$. Then by Lemma 1.2.8(i), A is a central p-subgroup of G, so that by Lemma 1.2.9, $\mathcal{U}(K(G/A))$ is hypercentral. Further, $(G/A)' = G'/A \cong Q_1 \times H$. So to derive a contradiction, it suffices to consider $G' = Q_1 \times H$. Next let m be the exponent of H, that is, $H^{p^m} = 1$. Set $B = \{x \in G' : x^{p^m} = 1\}$. Then $B \supseteq H$ and B is a finite characteristic subgroup of G'. Hence $B \lhd G$. Since B is a finite normal p-subgroup of G, $\mathcal{U}(K(G/B))$ is hypercentral as above with G'' replaced by B. But clearly $(G/B)' = G'/B \cong Z(p^\infty)$. Thus indeed it is enough to consider $G' \cong Z(p^\infty)$.

First we consider the case when p is odd. Since $G' \cong Z(p^\infty)$, for each $j \in \mathbb{N}$ choose $q_{2j+1} \in G'$ inductively such that $o(q_1) = p$ and $q_{2j+1}^p = q_{2j-1}$. We now construct a sequence $(g_n) \subseteq G$ with the following properties:

(1) $g_1, g_2 \in G$ with $b_1 = (g_2, g_1) \notin \langle q_1 \rangle$; and

(2) $g_{2n+1}, g_{2n+2} \in C_G\{g_1, g_2, \ldots, g_{2n}\}$ with $b_{2n+1} = (g_{2n+2}, g_{2n+1}) \notin \langle b_{2n-1}, q_{2n+1} \rangle$, for all $n \geq 1$.

Indeed, since $G' \cong Z(p^\infty)$, G' is locally finite. Thus since G' is infinite, there exist infinitely many commutators (x, y), x and y in G. Since $\langle q_1 \rangle$ is finite, such g_1 and g_2 exist. Next suppose $g_1, g_2, \ldots, g_{2n}$ have already been chosen satisfying properties (1) and (2). Let $C = C_G\{g_1, g_2, \ldots, g_{2n}\}$. Then since G is FC, $[G : C] < \infty$ by Poincare's Theorem. Hence $|G'| = \infty$ implies that $|C'| = \infty$ by [T; 7.5]. Thus C has infinitely many commutators (x, y) with $x, y \in C$ since $C' \subseteq G'$ is locally finite. So now since $\langle b_{2n-1}, q_{2n+1} \rangle$ is finite, $g_{2n+1}, g_{2n+2} \in C$ exist, and such a sequence can be constructed.

We use this sequence in G to construct a new one in $\mathcal{U}(KG)$. For each $i \in \mathbb{N}$, set $e_{2i+1} = p - 1$ if there exists no integer $j(i)$ such that $o(b_{2j(i)+1}) = p^{i+1} = o(q_{2i+1})$. If such a $j(i)$ does exist, set $e_{2i+1} = p - 2$. Notice that, in the latter case, this $j(i)$ is necessarily unique and less than i by the construction of (g_n). Also as $b_{2j(i)+1} \in G' \cong Z(p^\infty)$, then $b_{2j(i)+1} = q_{2i+1}^{r_{2i+1}}$ for some integer r_{2i+1}, $p \nmid r_{2i+1}$.

We make the following claim:

(i) $\alpha_{2n+1} = \prod_{i=0}^{n} (1 - q_{2i+1})^{e_{2i+1}} \prod_{i=0}^{n-1} (1 - b_{2i+1}) \neq 0$; and

(ii) $(1 - (1 - q_1)^{e_1} g_1,\ 1 - (1 - q_3)^{e_3} g_2 g_3,\ \ldots,\ 1 - (1 - q_{2n+1})^{e_{2n+1}} g_{2n} g_{2n+1}) =$ $1 - (-1)^n \alpha_{2n+1} g_1 g_2 \cdots g_{2n+1}$, for all $n \geq 1$.

Assuming the claim can be established, we then see immediately that $\mathcal{U}(KG)$ is not hypercentral, a contradiction that implies G' is in fact finite as required.

Proof of claim. First recall by Lemma 1.2.8, $G' \cong Z(p^\infty)$ implies that $G' \subseteq \zeta(G)$. Since for $n \geq 1$, $\left((1-q_{2n+1})^{e_{2n+1}}\right)^{p^{n+1}} = \left(1-q_{2n+1}^{p^{n+1}}\right)^{e_{2n+1}} = 0$, it is clear that each $1-(1-q_{2n+1})^{e_{2n+1}}g_{2n}g_{2n+1} \in \mathcal{U}(KG)$. Also since necessarily $e_1 = p-1$, and thus $2e_1 \geq p$, we have $(1-(1-q_1)^{e_1}g_1)^{-1} = 1+(1-q_1)^{e_1}g_1$.

Now we show $\alpha_{2n+1} \neq 0$ for all $n \geq 1$. To this end, let $p^m = o(b_{2n-1})$. Then $m-1 \geq n$ since $b_{2n-1} \notin \langle q_{2n-1} \rangle$ and $o(q_{2n-1}) = p^n$. Let $I = \{i \mid e_{2i+1} = p-2 \text{ and } 0 \leq i \leq n-1\}$. Then by previous remarks, i lies in I precisely when there exists a unique $j(i)$ such that $0 \leq j(i) < i \leq n-1$ and $b_{2j(i)+1} = q_{2i+1}^{r_{2i+1}}$ with $p \nmid r_{2i+1}$. Moreover,

$$\begin{aligned} 1-b_{2j(i)+1} &= 1-q_{2i+1}^{r_{2i+1}} \\ &= (1-q_{2i+1})\left(1+q_{2i+1}+q_{2i+1}^2+\cdots+q_{2i+1}^{r_{2i+1}-1}\right). \end{aligned}$$

Also notice for $0 \leq i \leq m-1$, $o(q_{2i+1}) \mid p^m$, so that if $e_{2i+1} = p-2$, then $j(i) \leq n-1$ since $o(b_{2n-1}) = p^m$. Hence

$$\begin{aligned} \beta &= \prod_{i=0}^{m-1}(1-q_{2i+1})^{e_{2i+1}} \prod_{i=0}^{n-1}(1-b_{2i+1}) \\ &= \prod_{i=0}^{m-1}(1-q_{2i+1})^{p-1} \prod_{i\in I}\left(1+q_{2i+1}+q_{2i+1}^2+\cdots+q_{2i+1}^{r_{2i+1}-1}\right) \\ &= \Big(\prod_{i\in I} r_{2i+1}\Big)\hat{q}_{2m-1} \neq 0 \end{aligned}$$

since

$$\begin{aligned} \prod_{i=0}^{m-1}(1-q_{2i+1})^{p-1} &= \prod_{i=0}^{m-1}(1-q_{2m-1})^{p^{m-i-1}(p-1)} \\ &= (1-q_{2m-1})^{(p-1)\sum_{i=0}^{m-1}p^i} = (1-q_{2m-1})^{p^m-1} = \hat{q}_{2m-1} \end{aligned}$$

by Lemma 1.2.12. But since $m-1 \geq n$, α_{2n+1} divides β, so that $\alpha_{2n+1} \neq 0$ either.

It remains to show (ii). To see this, let us first prove $(1-q_{2n+1})^{e_{2n+1}}\alpha_{2n+1}=0$. Let I be as above. Then

$$(1-q_{2n+1})^{e_{2n+1}}\alpha_{2n+1}$$

$$=(1-q_{2n+1})^{2e_{2n+1}}\prod_{i=0}^{n-1}(1-q_{2i+1})^{p-1}\prod_{i\in I}\left(1+q_{2i+1}+q_{2i+1}^2+\cdots+q_{2i+1}^{r_{2i+1}-1}\right)\prod_{\substack{i=0\\ i\notin I}}^{n-1}(1-b_{2i+1}).$$

But as above $\prod_{i=0}^{n-1}(1-q_{2i+1})^{p-1}=\hat{q}_{2n-1}$. We have two cases to consider. If $e_{2n+1}=p-1$, then $2e_{2n+1}\geq p$, and hence $(1-q_{2n+1})^{2e_{2n+1}}=(1-q_{2n-1})\gamma$, say, annihilates $\hat{q}_{2n-1}$. If $e_{2n+1}=p-2$, then there exists $j(n)<n$ not in I (by its uniqueness) such that $b_{2j(n)+1}=q_{2n+1}^{r_{2n+1}}$ for some $p\nmid r_{2n+1}$. Then $(1-q_{2n+1})^{e_{2n+1}}\alpha_{2n+1}$ contains the factor

$$\hat{q}_{2n-1}(1-q_{2n+1})^{2e_{2n+1}}(1-b_{2j(n)+1})$$

$$=\hat{q}_{2n-1}(1-q_{2n+1})^{2(p-2)}(1-q_{2n+1})(1+q_{2n+1}+q_{2n+1}^2+\cdots+q_{2n+1}^{r_{2n+1}-1})=0$$

since $2p-3\geq p$ for $p\geq 3$. This concludes the proof that $(1-q_{2n+1})^{e_{2n+1}}\alpha_{2n+1}=0$.

We now prove (ii) by induction on $n\geq 1$. For $n=1$, we have

$$(1-(1-q_1)^{e_1}g_1,\ 1-(1-q_3)^{e_3}g_2g_3)$$

$$=(1-(1-q_1)^{e_1}g_1-(1-q_3)^{e_3}g_2g_3+(1-q_1)^{e_1}(1-q_3)^{e_3}g_1g_2g_3)\cdot$$

$$(1+(1-q_1)^{e_1}g_1)(1+(1-q_3)^{e_3}g_2g_3+(1-q_3)^{2e_3}g_2^2g_3^2+\dots)$$

$$=(1-(1-q_3)^{e_3}g_2g_3+(1-q_1)^{e_1}(1-q_3)^{e_3}(1-b_1)g_1g_2g_3)\cdot$$

$$(1+(1-q_3)^{e_3}g_2g_3+(1-q_3)^{2e_3}g_2^2g_3^2+\dots)$$

$$=1+(1-q_1)^{e_1}(1-q_3)^{e_3}(1-b_1)g_1g_2g_3,$$

as required, since $(1-q_3)^{e_3}\alpha_3 = 0$. Now consider $n > 1$. By induction we have

$$
\begin{aligned}
&(1-(1-q_1)^{e_1}g_1,\ 1-(1-q_3)^{e_3}g_2g_3,\ldots,1-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3})\\
&\quad=(1-(-1)^n\alpha_{2n+1}g_1g_2\cdots g_{2n+1},\ 1-(1-q_{2n+3})^{e_{2n+2}}g_{2n+2}g_{2n+3})\\
&\quad=(1-(-1)^n\alpha_{2n+1}g_1g_2\cdots g_{2n+1}-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3}\\
&\qquad+(-1)^n(1-q_{2n+3})^{e_{2n+3}}\alpha_{2n+1}g_1g_2\cdots g_{2n+3})\cdot\\
&\qquad(1+(-1)^n\alpha_{2n+1}g_1g_2\cdots g_{2n+1})\cdot(1-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3})^{-1}\\
&\quad=(1-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3}\\
&\qquad+(-1)^n(1-q_{2n+3})^{e_{2n+3}}\alpha_{2n+1}g_1g_2\cdots g_{2n+3}\\
&\qquad-(-1)^n(1-q_{2n+3})^{e_{2n+3}}\alpha_{2n+1}g_{2n+2}g_{2n+3}g_1g_2\cdots g_{2n+1})\cdot\\
&\qquad(1-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3})^{-1}\\
&\quad=(1-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3}\\
&\qquad+(-1)^n(1-q_{2n+3})^{e_{2n+3}}\alpha_{2n+1}(1-b_{2n+1})g_1g_2\cdots g_{2n+3})\cdot\\
&\qquad(1+(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3}+(1-q_{2n+3})^{2e_{2n+3}}g_{2n+2}^2g_{2n+3}^2+\cdots)\\
&\quad=1-(-1)^{n+1}\alpha_{2n+3}g_1g_2\cdots g_{2n+3}
\end{aligned}
$$

since $\alpha_{2n+1}^2 = 0$ and $(1-q_{2n+3})^{e_{2n+3}}\alpha_{2n+3} = 0$. This concludes the proof of the proposition for the odd p case.

Finally consider when $p = 2$. Recall that we can assume $G' \cong Z(2^\infty)$. Let $q \in G'$ be such that $o(q) = 2$. Construct a sequence (g_n) in G as follows:

(1) $g_1, g_2 \in G$ with $b_1 = (g_2, g_1) \notin \langle q\rangle$; and

(2) $g_{2n+1}, g_{2n+2} \in C_G\{g_1, g_2, \ldots, g_{2n}\}$ with $b_{2n+1} = (g_{2n+2}, g_{2n+1}) \notin \langle b_{2n-1}\rangle^{1/2}$.

In a similar fashion to the odd case, this can be done by using the fact that G is an FC-group implies that G' is locally finite, so that G and hence $C_G\{g_1, g_2, \ldots, g_{2n}\}$ contain infinitely many commutators, while $\langle b_{2n-1}\rangle^{1/2} \subseteq Z(2^\infty)$ is finite.

We construct another sequence (q_{2n+1}) in G'. Let $q_1 = q$. For $n \geq 1$, pick $q_{2n+1} \in G'$ such that $q_{2n+1}^2 = q_{2n-1}$ unless there exists a $j(n) \in \mathbf{N}$ such that $o(b_{2j(n)-1}) = 2o(q_{2n-1})$, in which case choose q_{2n+1} such that $q_{2n+1}^4 = q_{2n-1}$. Observe that if $o(b_{2j(n)-1}) = 2o(q_{2n-1})$, then $o(b_{2j(n)+1}) \neq 4o(q_{2n-1})$ for otherwise $b_{2j(n)+1} \in \langle b_{2j(n)-1}\rangle^{1/2}$, contrary to (2). Thus for each $n \geq 1$ there is precisely one "q" or "b" with order 2^n. Also (1) and (2) imply $2o(q_{2i+1}) \leq o(b_{2j+1})$ and $4o(b_{2i+1}) \leq o(b_{2j+1})$ for all $i < j$.

We claim now that, for all $n \geq 1$,

(i) $\alpha_{2n+1} = \prod_{i=0}^{n}(1 + q_{2i+1}) \prod_{i=0}^{n-1}(1 + b_{2i+1}) \neq 0$ and

(ii) $(1+(1+q_1)g_1,\ 1+(1+q_3)g_2g_3, \ldots, 1+(1+q_{2n+1})g_{2n}g_{2n+1}) = 1+\alpha_{2n+1}g_1 \cdots g_{2n+1}$.

Obviously, this would contradict that $\mathcal{U}(KG)$ is hypercentral, and so G' would be finite.

To prove the claim, first observe that if m is minimal such that $2o(b_{2n-1}) \leq o(q_{2m+1})$, then $\beta = \prod_{i=0}^{m}(1 + q_{2i+1}) \prod_{i=0}^{n-1}(1 + b_{2i+1})$ is a nonzero scalar multiple of $\hat{q}_{2m+1}$. Indeed q_{2m+1} is the most primitive element appearing in the expression for β, and by construction precisely one element of each lesser order also appears. Then if x is one of the q's or b's appearing in the expression, $1 + x = 1 + q_{2m+1}^{2^j r}$ for some integers r and j, where r is odd and j is unique to x. Hence $1 + x = (1 + q_{2m+1})^{2^j}(1+q_{2m+1}^{2^j}+\cdots+q_{2m+1}^{2^j(r-1)})$. Now reasoning as in the odd case, Lemma 1.2.12 gives us $\beta = s\hat{q}_{2m+1} \neq 0$ for some integer s. In particular, since by the remarks above $m \geq n$, we see that α_{2n+1} divides β and thus cannot be zero.

It remains to prove (ii). To do this, we first show that $(1+q_{2n+1})\alpha_{2n+1} = 0$. Let m be maximal such that $2o(b_{2m-1}) \leq o(q_{2n+1})$, and let $\gamma = \prod_{i=0}^{n}(1 + q_{2i+1}) \prod_{i=0}^{m-1}(1 +$

b_{2i+1}), where the latter product is considered to be 1 if no such m exists. Then $\gamma = s\hat{q}_{2n+1}$ for some integer s by precisely the same arguments used on β above. Since necessarily $m \leq n$, γ divides α_{2n+1}. Thus since $(1 + q_{2n+1})\gamma = 0$, we see $(1 + q_{2n+1})\alpha_{2n+1} = 0$. An easy induction using this fact proves (ii).

This finishes the proof of the proposition.

□

Finally, notice that the proof of Theorem 1.2.1 (i) ⇒ (iii) now follows at once by combining the results of Lemma 1.2.5 and Propositions 1.2.10 and 1.2.13.

1.3 The characteristic zero case.

We now examine the situation when K ia field of characteristic zero.

1.3.1 THEOREM. *Suppose K is a field of characteristic zero and G is any group. Then $\mathcal{U}(KG)$ is hypercentral if and only if G is hypercentral and the torsion subgroup of G is central.*

The proof of this theorem is basically a careful reworking of the proof appearing in [S; VI] of the classification of group algebras KG having the property that $\mathcal{U}(KG)$ is nilpotent, when K is a field of characteristic zero.

In addition to notation previously introduced, the following will also be used. Denote by $T(G)$ the torsion subgroup of G; let Cr denote an unrestricted direct product; $\dot{F}$ is the multiplicative group of a field F; and by $R(G, \rho, \sigma)$ we mean the crossed product of a group G over a ring with identity R with factor system ρ and automorphisms σ (see [S] for details). *For the remainder of this chapter, K denotes a field of characteristic zero.*

We will need some preliminary results

1.3.2 LEMMA. *Let G be a torsion free locally nilpotent group. Then the units of any crossed product $K(G,\rho,\sigma)$ are trivial.*

Proof. The same proof as [S; VI.1.7] works here since we may assume G is finitely generated.

□

1.3.3 LEMMA. *Suppose $\mathcal{U}(KG)$ is hypercentral. Then we have*

(i) $T(G)$ is abelian and every subgroup of $T(G)$ is normal in G;

(ii) every idempotent of KG is central.

Proof. (i) Notice $\mathcal{U}(\mathbf{Q}K_8)$ is not hypercentral, where K_8 is the quaternion group of order 8. Indeed by [S; VI.1.14], $\mathbf{Q}K_8 \simeq \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{H}$ where $\mathbf{H}$ is the rational quaternion algebra. But $\mathcal{U}(\mathbf{H})/\dot{\mathbf{Q}}$ has no centre and thus is not hypercentral. Using this additional fact, the proof follows exactly as for [S; VI.3.7].

(ii) The proof of [S; VI.3.13] works here with part (i) in place of [S; VI.3.7] if we notice as well that if G is finitely generated, then $T(G)$ is finite since $G < \mathcal{U}(KG)$ implies that G is locally nilpotent. □

Now we are ready to prove the theorem. First let us deal with necessity; so we assume that $\mathcal{U}(KG)$ is hypercentral. Then of course G too is hypercentral. We would like to show $T = T(G)$ is central. By Lemma 1.3.3, T is abelian, so if this is not the case, we can find $t \in T$ and $x \in G$ such that $xt \neq tx$ and $o(x) = \infty$. Let α denote the automorphism of $\mathbf{Q}\langle t \rangle$ induced by conjugation by x. Also by Lemma 1.3.3, every subgroup of T is normal in G and every idempotent of KG is central. From this we may write $\mathbf{Q}\langle t \rangle = \mathrm{Dr}_{i=1}^{k} F_i$, where the F_i are cyclotomic extensions of $\mathbf{Q}$ [S; II.2.6]. Also $t^\alpha = xtx^{-1} \in \langle t \rangle$ and $F_i^\alpha = F_i$ for all i, so that α may be regarded as an automorphism of F_i with finite order s, say. Since $xt \neq tx$,

we can assume without loss that α does not act as the identity on F_1. By an easy induction, it can be shown that if $f \in \dot{F}_1$, then $(f, x, x, \ldots, x) = f^{(1-\alpha)^m}$, where the x appears m times. Then since $\mathcal{U}(KG)$ is hypercentral, for each $f \in \dot{F}_1$, there exists an integer m depending on f such that $f^{(1-\alpha)^m} = 1$. Recall that we already know $f^{(1-\alpha^s)} = 1$ for all $f \in \dot{F}_1$. Now since there exist polynomials $u(X),\ v(X) \in \mathbf{Z}[X]$ and $0 \neq a \in \mathbf{Z}$ such that $u(X)(1-X)^m + v(X)(1-X^s) = a(1-X)$, we have that $u(\alpha)(1-\alpha)^m + v(\alpha)(1-\alpha^s) = a(1-\alpha)$. It follows that for each $f \in \dot{F}_1$ there is an $0 \neq a \in \mathbf{Z}$ depending on f such that $f^{a(1-\alpha)} = 1$. In other words $(f^a)^\alpha = f^a$. Denote by E_1 the fixed field of α in F_1. We have just shown for each $f \in F_1$, some power of f lies in E_1. Now since α does not act as the identity on F_1, let f be a fixed element of $F_1 \backslash E_1$. For each $r \in E_1$, choose an $a(r) \in \mathbf{Z}^+$ such that $(f+r)^{a(r)} \in E_1$. Then $(f+r)^{a(r)} = ((f+r)^{a(r)})^\alpha = ((f+r)^\alpha)^{a(r)}$, so that $(f+r)^\alpha = (f+r)\xi_{a(r)}$, where $\xi_{a(r)} \in F_1$ is an $a(r)$th root of unity. Since F_1 is a cyclotomic extension of $\mathbf{Q}$, F_1 has only finitely many roots of unity. Thus since E_1 is infinite, for infinitely many $r \in E_1$, $(f+r)^\alpha = (f+r)\xi$ for a fixed root of unity ξ. But $(f+r)\xi = (f+r)^\alpha = f^\alpha + r = f\xi_{a(0)} + r$, so that $r(\xi - 1) = f(\xi_{a(0)} - \xi)$. Since this holds for infinitely many $r \in E_1$, we can conclude $\xi - 1 = \xi_{a(0)} - \xi$. Thus $f^\alpha = f$, which contradicts our choice of f. Hence T is indeed central.

Finally we turn to sufficiency. Assume that G is hypercentral of class γ, that is $G = \zeta_\gamma(G)$, and $T \subseteq \zeta(G)$. We intend to show that $\mathcal{U}(KG)$ is hypercentral of class at most $\gamma + 1$. Since T is abelian and $char(K) = 0$, by [P1; 7.4.2], KT is semisimple. Then by [He; 2.2.1], KT is a subdirect product of fields. Thus let $KT \xrightarrow{j} \underset{i \in I}{Cr}\ F_i$ be a ring monomorphism such that $\pi_i j(KT) = F_i$ for each $i \in I$, where the F_i are fields and the $\pi_i : \underset{i \in I}{Cr}\ F_i \to F_i$ are the canonical epimorphisms. We also have $KG = (KT)(G/T, \rho, \sigma)$, a crossed product of G/T over KT with natural ρ and σ. Here σ is trivial since KT is central in KG. Since $j\mathcal{U}(KT) \subseteq U(\underset{i \in I}{Cr}\ F_i)$, j

induces a monomorphism $KG = (KT)(G/T, \rho, 1) \overset{j}{\hookrightarrow} (\underset{i\in I}{Cr}\ F_i)(G/T, \hat{\rho}, 1)$ where $\hat{\rho}$ acts like ρ. Which is to say, if $\bar{g}\bar{h} = \rho_{g,h}\overline{gh}$ in $(KT)(G/T, \rho, 1)$, then $\bar{g}\bar{h} = \hat{\rho}_{g,h}\overline{gh}$ in $(\underset{i\in I}{Cr}\ F_i)(G/T, \hat{\rho}, 1)$, where $\hat{p}_{g,h} = j(\rho_{g,h})$. Because elements of $\underset{i\in I}{Cr}\ F_i$ commute with G/T, we also have that the obvious map

$$(\underset{i\in I}{Cr}\ F_i)(G/T, \hat{\rho}, 1) \longrightarrow \underset{i\in I}{Cr}\ F_i(G/T, \hat{\rho}_i, 1),$$

where the $\hat{\rho}_i$ are the natural projections of $\hat{p}$, is an isomorphism. Thus we may consider $KG \hookrightarrow \underset{i\in I}{Cr}\ F_i(G/T, \hat{\rho}_i, 1)$. Now $\mathcal{U}(KG) \hookrightarrow \underset{i\in I}{Cr}(\dot{F}_i \cdot G/T)$ by Lemma 1.3.2. Since each $\dot{F}_i$ and G/T commute, clearly each $\dot{F}_i \cdot (G/T)$ is of class $\leq \gamma + 1$. It now follows easily that $\mathcal{U}(KG)$ has class at most $\gamma + 1$. This concludes the proof of the theorem.

□

Chapter Two

RESTRICTED LIE DIMENSION SUBGROUPS

2.1 Introduction.

Let $\Delta_R(G)$ be the augmentation ideal of a group ring RG. The mth restricted Lie ideal $\Delta_R^{[m]}(G)$ of $\Delta_R(G)$, $m \geq 1$, is the (associative) ideal generated by all the Lie commutators $[x_1, x_2, \ldots, x_m]$, $x_i \in \Delta_R(G)$, where $[x_1] = x_1$, $[x_1, x_2] = x_1x_2 - x_2x_1$ and, inductively, $[x_1, x_2, \ldots, x_m] = [[x_1, x_2, \ldots, x_{m-1}], x_m]$. There is a related sequence of Lie powers $\Delta_R^{(m)}(G)$ of $\Delta_R(G)$ defined inductively by $\Delta_R^{(1)}(G) = \Delta_R(G)$ and $\Delta_R^{(m+1)}(G) = [\Delta_R^{(m)}(G), \Delta_R(G)]RG$, the ideal generated by the Lie products $[x, y]$, $x \in \Delta_R^{(m)}(G)$ and $y \in \Delta_R(G)$. It is clear that in general we have $\Delta_R^{[m]}(G) \subseteq \Delta_R^{(m)}(G)$; however, equality need not hold [GL].

Let $D_{(m),R}(G)$ denote the mth Lie dimension subgroup of G, while $D_{[m],R}(G)$ denotes the mth restricted Lie dimension subgroup; namely,

$$D_{(m),R}(G) = G \cap (1 + \Delta_R^{(m)}(G)) \text{ and } D_{[m],R}(G) = G \cap (1 + \Delta_R^{[m]}(G)).$$

The principal objective of this chapter is to characterize the subgroups $D_{[m],K}(G)$ when K is a field. These subgroups arise in connection with Jennings' unsettled conjecture that $\Delta_K^{[m]}(G) = 0$ if and only if $\mathcal{V}_m(KG) = 1$, where $\mathcal{V}(KG)$ is the groups of units of augmentation one in KG and $\mathcal{V}_m(KG)$ is the mth term of the lower central series of $\mathcal{V}(KG)$. In fact, it turns out that, for all $m \geq 1$, $D_{[m],K}(G)$ coincides with $D_{(m),K}(G)$ when K has characteristic $p \neq 2, 3$. Since these latter group already have been given a precise description by Passi and Sehgal in [PS2], this will complete our task.

Before we begin, let us make some more comments about notation. Let G_i denote the ith term of the lower central series of a group G, while G_i^n will be the

subgroup of G generated by the elements of the form x^n, where $x \in G_i$. Finally, $\sqrt{G_i}$ will denote the isolator of G_i in G.

The main result of this chapter is as follows.

2.1.1 THEOREM. *(i) If K is a field of characteristic $p \geq 5$, then*

$$D_{[m],K}(G) = D_{(m),K}(G) = \prod_{(i-1)p^j \geq m-1} G_i^{p^j} \quad \text{for all } m \geq 1.$$

(ii) If K is a field of characteristic zero, then

$$D_{[m],K}(G) = D_{(m),K}(G) = G_2 \cap \sqrt{G_m} \quad \text{for all } m \geq 2.$$

2.2 Characteristic $p \geq 5$.

Some of our initial results hold for a more general class of rings than group rings, and may be of some independent interest. Let A be any associative unitary ring. Define $L_m(A)$ to be the associative ideal of A generated by all the Lie commutators $[x_1, x_2, \ldots, x_m]$, $x_i \in A$, where the Lie operation is given by $[x, y] = xy - yx$. Then it is easy see that $L_m(RG)$ is merely $\Delta_R^{[m]}(G)$ when $m \geq 2$.

To distinguish between Lie and group commutators, let us denote group commutators with round brackets. Given an associative ring A, let $\mathcal{U}(A)$ be its group of units and define for each integer $i \geq 1$ the set

$$\mathcal{C}_i(A) = \left\{(u_1, u_2, \ldots, u_k)^{\pm 1} : u_1, u_2, \ldots, u_k \in \mathcal{U}(A) \text{ and } k \geq i\right\}.$$

Notice that $\mathcal{C}_i(A)$ is normal in $\mathcal{U}(A)$. Indeed, for any $v \in \mathcal{U}(A)$, $(u_1, u_2, \ldots, u_k)^v = (u_1^v, u_2^v, \ldots, u_k^v) \in \mathcal{C}_i(A)$.

We will make use of the following fundamental result of Gupta and Levin.

2.2.1 LEMMA. *For each $m \geq 1$, we have $\mathcal{U}_m(A) \subseteq 1 + L_m(A)$.*

Proof. [GL; Theorem A]

Now we make a minor adjustment to a recent result of Sharma and Srivastava.

2.2.2 LEMMA. *If A is any associative ring in which both 2 and 3 are units, then $(c-1)^n \in L_{(i-1)n+1}(A)$ for all $n \geq 0$ and $c \in \mathcal{C}_i(A)$.*

Proof. [SS; Lemma 2.11] tells us that $((u_1, u_2, \ldots, u_k) - 1)^n \in L_{(k-1)n+1}(A) \subseteq L_{(i-1)n+1}(A)$ whenever $u_1, u_2, \ldots, u_k \in \mathcal{U}(A)$. Thus from the identity $(x^{-1} - 1)^n = (-1)^n x^{-n}(x-1)^n$ and the fact that $L_{(i-1)n+1}(A)$ is an ideal of A, we also have $((u_1, u_2, \ldots, u_k)^{-1} - 1)^n \in L_{(i-1)n+1}(A)$.

□

2.2.3 THEOREM. *Suppose A is an associative ring of prime characteristic $p \geq 5$ in which both 2 and 3 are invertible. Then we have the following:*

$$(\mathcal{U}_i(A) - 1)^{p^j} \subseteq L_m(A) \text{ whenever } (i-1)p^j \geq m - 1.$$

Proof. If $i = 1$ then $m = 1$, and so the inclusion is trivial in this case. For each $i \geq 2$ choose the minimal nonnegative integer j_i such that $(i-1)p^{j_i} \geq m - 1$. Then of course $j_i = 0$ for $i \geq m$. We prove by induction on k, $0 \leq k \leq m-2$, that $\mathcal{U}_{m-k}^{p^{j_{m-k}}} \subseteq 1 + L_m$. Here $\mathcal{U}_{m-k} = \mathcal{U}_{m-k}(RG)$ and $L_m = L_m(A)$. When $k = 0$, the inclusion follows from Lemma 2.2.1. Suppose then that $\mathcal{U}_i^{p^{j_i}} \subseteq 1 + L_m$ for all $i > m - k$. We must demonstrate that this implies $\mathcal{U}_{m-k}^{p^{j_{m-k}}} \subseteq 1 + L_m$. Abbreviate j_{m-k} by j. Since $\mathcal{U}_{m-k}$ is the normal closure of the subgroup generated by the $(m-k)$–fold commutators of $\mathcal{U}$, it suffices for us to show that if $x_1, x_2, \ldots, x_r \in \mathcal{C}_{m-k}$, then $(x_1 x_2 \cdots x_r)^{p^j} \in 1 + L_m$. To do this, we use the collection process of P. Hall [H;

Theorem 12.3.1]. Collecting all the basic commutators of weight less than $\frac{m}{m-k}$ in the $x's$ we get

$$(x_1x_2\cdots x_r)^{p^j} = x_1^{p^j}x_2^{p^j}\cdots x_r^{p^j}c_{r+1}^{e_{r+1}}c_{r+2}^{e_{r+2}}\cdots c_s^{e_s}R,$$

where R lies in $\mathcal{U}_m$ and $c_{r+1},\ldots,c_s$ are basic commutators of weights $w_{r+1},\ldots,w_s$ in the $x's$, respectively. The exponents e_l, $r+1 \le l \le s$, are of the form $b_1\binom{p^j}{1} + \cdots + b_{w_l}\binom{p^j}{w_l}$, where the $b's$ are nonnegative integers. Note that by the choice of $j = j_{m-k}$,

$$w_l < \frac{m}{m-k} < \frac{m-1}{(m-k)-1} \le p^j.$$

By Lemma 2.2.2 it remains only to prove that $c_l^{e_l} \in 1+L_m$ for every l, $r+1 \le l \le s$. Since c_l has weight w_l in the $x's$, which all lie in $\mathcal{U}_{m-k}$, $c_l^{e_l} \in \mathcal{U}_{(m-k)w_l}^{e_l}$. Let $|n|_p$ denote the p-part of an integer n. Choose an integer r, $1 \le r \le w_l$, such that $\left|\binom{p^j}{r}\right|_p$ is minimal. It is well known [P1; p.482] that $\frac{p^j}{|r|_p}\left|\left|\binom{p^j}{r}\right|_p\right.$, so that $\frac{p^j}{|r|_p}\Big|e_l$. Thus $c_l^{e_l} \in \mathcal{U}_{(m-k)w_l}^{\frac{p^j}{|r|_p}}$. But since $w_l \ge 2$, we have

$$((m-k)w_l - 1)\frac{p^j}{|r|_p} \ge ((m-k)w_l - 1)\frac{p^j}{w_l} \ge ((m-k)-1)p^j \ge m-1$$

by our choice of j. Hence $p^{j(m-k)w_l}\left|\frac{p^j}{|r|_p}\right.$. Finally since $(m-k)w_l > m-k$, by the induction hypothesis we have

$$c_l^{e_l} \in \mathcal{U}_{(m-k)w_l}^{\frac{p^j}{|r|_p}} \subseteq \mathcal{U}_{(m-k)w_l}^{p^{j(m-k)w_l}} \subseteq 1+L_m$$

as required.

□

We can immediately obtain the following result, which may have some implications to Jennings' conjecture.

2.2.4 COROLLARY. *If R is a commutative ring of prime characteristic $p \geq 5$ in which both 2 and 3 are invertible, then for any group G we have that*

$$\Delta_R^{[m]}(G) = 0 \text{ implies that } \prod_{(i-1)p^j \geq m-1} \mathcal{V}_i(RG)^{p^j} = 1,$$

for all $m \geq 1$.

Passi and Sehgal classified Lie dimension subgroups in the following way.

2.2.5 LEMMA. *If K is a field of characteristic $p > 0$, then*

$$D_{(m),K}(G) = \prod_{(i-1)p^j \geq m-1} G_i^{p^j} \quad \text{for all } m \geq 1.$$

Proof. [PS2; Theorem 4.10].

Proof of Theorem 2.1.1(i). Trivially $D_{[m],K}(G) \subseteq D_{(m),K}(G)$, so that by Lemma 2.2.5 it suffices to show $G_i^{p^j} \subseteq D_{[m],K}(G)$ whenever $(i-1)p^j \geq m-1$. But $G_i^{p^j} - 1 = (G_i - 1)^{p^j} \subseteq \Delta_K^{[m]}(G)$ by Theorem 2.2.3.

□

2.3 Characteristic zero.

The second half of Theorem 2.1.1 follows more easily using previous work done in the area. First we will need the other half of the above theorem of Passi and Sehgal.

2.3.1 LEMMA. *If K is a field of characteristic zero, then*

$$D_{(m),K}(G) = G_2 \cap \sqrt{G_m} \quad \text{for all } m \geq 2.$$

Proof. See [PS2; Theorem 3.2].

Now we recall a theorem of Jennings'.

2.3.2 LEMMA. *$\Delta_R^{[2]}(G)/\Delta_R^{[m]}(G)$ is nil for all $m \geq 2$.*

Proof. See [J].

Proof of Theorem 2.1.1(ii). Certainly $D_{[m],K}(G) \subseteq D_{(m),K}(G)$, so that by Lemma 2.3.1 we need only show that $G_2 \cap \sqrt{G_m} \subseteq D_{[m],K}(G)$.

Let $x \in G_2 \cap \sqrt{G_m}$. Then $x^r \in G_m$ for some $r > 0$. From the identity

$$x^r - 1 = \binom{r}{1}(x-1) + \binom{r}{2}(x-1)^2 + \cdots + \binom{r}{r}(x-1)^r,$$

we see by Lemma 2.2.1 that

$$x - 1 \equiv -\frac{1}{r}\left[\binom{r}{2}(x-1)^2 + \cdots + \binom{r}{r}(x-1)^r\right] \tag{2.3.3}$$

modulo $\Delta_K^{[m]}(G)$. Now since $x-1 \in \Delta_K^{(2)}(G)$, by Lemma 2.3.2, repeated substitution of equation 2.3.3 into itself yields that $x - 1 \in \Delta_K^{[m]}(G)$.

□

2.4 Residual Lie nilpotence.

A classification of when a group algebra KG is residually Lie nilpotent, in other words $\bigcap_{m\geq 1} \Delta_K^{(m)}(G) = 0$, has been given by Parmenter, Passi and Sehgal in [PPS]. Using this classification and Theorem 2.1.1, we can immediately determine exactly when KG is residually Lie nilpotent in the restricted sense -at least when the characteristic of the field K is different from 2 and 3.

2.4.1 COROLLARY. *Let G be a group, and K any field of characteristic $p \neq 2, 3$. Then the following are equivalent:*

(i) $\bigcap_{m\geq 1} \Delta_K^{(m)}(G) = 0$;

(ii) $\bigcap_{m\geq 1} \Delta_K^{[m]}(G) = 0$;

(iii) $\bigcap_{m\geq 1} D_{[m],K}(G) = 1$;

(iv) $\bigcap_{m\geq 1} D_{(m),K}(G) = 1$; *and*

(v) either $p = 0$ *and* G *is residually 'nilpotent with derived subgroup torsion-free', or* $p \geq 5$ *and* G *is residually 'nilpotent with derived subgroup a* $p-$*group of bounded exponent'.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial given Theorem 2.1.1, while (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) is [PPS; 7.4].

□

2.5 The exceptional characteristics $p = 2, 3$.

The question of whether or not $D_{[m],K}(G) = D_{(m),K}(G)$ when the field K has one of the exceptional characteristics of $p = 2$ or 3 appears to be more difficult to settle. We show here that at least for $m \leq 6$, these two subgroups of G coincide.

In this section K will always denote a field of characteristic $p = 2$ or 3.

2.5.1 LEMMA. *To prove* $D_{[m],K}(G) = D_{(m),K}(G)$, *we may assume that* $D_{[m],K}(G) = 1$, *and prove that this implies* $D_{(m),K}(G) = 1$.

Proof. We know that $D_{[m],K}(G) \subseteq D_{(m),K}(G)$. Let $^-: KG \to K(G/D_{[m],K}(G))$ denote the canonical epimorphism. If $\overline{x} \in \frac{D_{(m),K}(G)}{D_{[m],K}(G)}$, then $x - 1 \in \Delta_K^{(m)}(G)$, so that $\overline{x} - 1 \in \Delta_K^{(m)}(\overline{G})$ and hence $\overline{x} \in D_{(m),K}(\overline{G})$. This shows that $\frac{D_{(m),K}(G)}{D_{[m],K}(G)} \subseteq D_{(m),K}(G/D_{[m],K}(G))$. Consequently, $D_{(m),K}(G/D_{[m],K}(G)) = 1$ implies that $\frac{D_{(m),K}(G)}{D_{[m],K}(G)} = 1$, so that $D_{[m],K}(G) = D_{(m),K}(G)$. This proves the lemma.

□

The following lemma is a collection results of Gupta, Levin and Sehgal taken from [LS] and [GL].

2.5.2 LEMMA: *(i) For* $x, y \in G$, $((x,y)-1)^n \in \Delta_K^{[n+1]}(G)$.

(ii) For $x, y, z \in G$, $((x,y,z)-1)^2 \in \Delta_K^{[5]}(G)$.

(iii) For all $m, n > 1$, $(\Delta_K^{[m]}(G))(\Delta_K^{[n]}(G)) \subseteq \Delta_K^{[m+n-2]}(G)$.

2.5.3 PROPOSITION. *If* K *is any field and* G *is a group with nilpotence class two, then* $D_{[m],K}(G) = D_{(m),K}(G)$ *for all* $m \geq 1$.

Proof. From Theorem 2.1.1(ii), we can assume that the characteristic of K is positive. By Lemma 2.5.1, we can assume that $D_{[m],K}(G) = 1$. Let j be minimal such that $p^j \geq m-1$. Since $G_3 = 1$, then $G_2 \subseteq \zeta(G)$. Hence G_2 is abelian, so that $G_2^{p^j}$ is generated by $\{(x,y)^{p^j} : x, y \in G\}$. Now since Lemma 2.5.2 tells us that $(x,y)^{p^j} - 1 = ((x,y)-1)^{p^j} \in \Delta_K^{[p^j+1]}(G) \subseteq \Delta_K^{[m]}(G)$, we have $G_2^{p^j} \subseteq D_{[p^j+1],K}(G) \subseteq D_{[m],K}(G) = 1$. Since here $D_{(m),K}(G) = G_2^{p^j}$, we are done by Lemma 2.5.1.

□

2.5.4 COROLLARY. $D_{[3],K}(G) = D_{(3),K}(G)$ *for arbitrary groups* G *and fields* K.

Proof. We can assume $D_{[3],K}(G) = 1$ by Lemma 2.5.1, so that $G_3 = 1$ by Lemma 2.2.1. Now we can apply Proposition 2.5.3 to obtain the result.

□

Using Lemma 2.5.2, we can stretch these ideas a bit further to obtain the following proposition.

2.5.5 PROPOSITION. *If* K *is any field, then* $D_{[m],K}(G) = D_{(m),K}(G)$ *for all* $m \leq 6$.

Proof. Of course we may assume that the characteristic of K is either 2 or 3. Let us prove the case $m = 6$, the other cases being similar. As usual, assume that $D_{[6],K}(G) = 1$. We will need the following well known fact, which is proved easily by induction. If x, y lie in a group H, then modulo H_3 we have

$$(xy)^n \equiv x^n y^n (y,x)^{\frac{n(n-1)}{2}}.$$

First assume $p = 3$. Then $D_{(6),K}(G) = G_2^9 G_3^3 G_6$. Already, of course, $G_6 = 1$. With $H = G_2$ above, we see easily that G_2^9 is generated by $\{(x,y)^9 : x, y \in G\}$. But by Lemma 2.5.2 we have $(x,y)^9 - 1 = ((x,y) - 1)^9 \subseteq \Delta_K^{[9+1]}(G) \subseteq \Delta_K^{[6]}(G)$, and subsequently $G_2^9 = 1$. Since G_3 is abelian, G_3^3 is the normal closure of the group generated $\{(x,y,z)^3 : x, y, z \in G\}$. However by Lemma 2.5.2 we have $((x,y,z) - 1)^3 = ((x,y,z) - 1)^2((x,y,z) - 1) \in \Delta_K^{[5]}(G)\Delta_K^{[3]}(G) \subseteq \Delta_K^{[6]}(G)$. This implies that $G_3^3 = 1$, and so $D_{(6),K}(G) = 1$.

Now assume that $p = 2$. Then $D_{(6),K}(G) = G_2^8 G_3^4 G_4^2 G_6$. Suppose x and $y = (a,b)$ are commutators in G. Then using Lemma 2.5.2 along with the "fact" above with $H = G_2$, we have

$$(xy)^8 = x^8 y^8 (y,x)^{28} = x^8 y^8 ((a,b,x)^4)^7 \in D_{[5+1],K}(G) D_{[5+5-2],K}(G) = 1.$$

Hence $G_2^8 = 1$. Next since G_3 is abelian, we know that G_3^4 is the normal closure of the group generated by $\{(x,y,z)^4 : x, y, z \in G\}$. But by Lemma 2.5.2, $((x,y,z) - 1)^4 \in D_{[5+5-2],K}(G) = 1$, so that $G_3^4 = 1$. Finally, because G_4 is abelian, G_4^2 is the normal closure of the group generated by $\{(x,y,z,w)^2 : x, y, z, w \in G\}$. But by Lemma 2.5.2, $((x,y,z,w) - 1)^2 \in D_{[4+4-2],K}(G) = 1$, so that $G_4^2 = 1$. This completes the proof.

□

Chapter Three

GENERALIZED DIMENSION SUBGROUPS

3.1 Introduction.

Let G be a group and let R be any commutative ring with unity. If N is a normal subgroup of G, let $\Delta_R(G,N)$ denote the kernel of canonical epimorphism $RG \to R(G/N)$. Now for each pair $m \geq 1, k \geq 0$ define the ideal $\Delta^R_{m,k}(G) = \sum \prod_j \Delta_R(G, G_{m_j})$, where the sum is over all sequences $\{m_j\}$, for which $m_j \geq 1$ and $\sum(m_j - k) = m - k$. Notice that when $k = 0$ we have that $\Delta^R_{m,0}(G)$ is merely $\Delta^m_R(G)$. For $k = 1$, consider the following very useful formula of Sandling [Sa1; 3.9].

3.1.1 LEMMA. *For each $m \geq 1$, $\Delta^R_{m,1}(G) = \Delta^{(m)}_R(G)$, the m-th Lie power of $\Delta_R(G)$. (See section 2.1 for the definition of $\Delta^{(m)}_R(G)$.)*

Set $E^R_{m,k}(G) = G \cap (1 + \Delta^R_{m,k}(G))$. Therefore, $E^R_{m,0}(G) = D^R_m(G) = G \cap (1 + \Delta^m_R(G))$, the m-th (ordinary) dimension subgroup of G over R, while $E^R_{m,1}(G) = D^R_{(m)}(G) = G \cap (1 + \Delta^{(m)}_R(G))$, the m-th Lie dimension subgroup of G over R. Thus, in some sense, we have a bridge between $D^R_m(G)$ and G_m, the m-th term of the lower central series of G:

$$D^R_m(G) = E^R_{m,0}(G) \supseteq D^R_{(m)}(G) = E^R_{m,1}(G) \supseteq E^R_{m,2}(G) \supseteq \cdots \supseteq E^R_{m,m-1}(G) = G_m.$$

These $E^R_{m,k}(G)$ are the "generalized dimension subgroups" of G.

Assign to G the subgroups $F_{m,k}(G) = \prod_{(i-k)p^j \geq m} {G_i}^{p^j}$ for each pair of integers $m \geq 1$ and $k \geq 0$. Here ${G_i}^{p^j}$ is the subgroup of G generated by $\{x^{p^j} : x \in G_i\}$. In Lazard's fundamental paper [L] he showed that the m-th dimension subgroup of G over a field K of characteristic $p > 0$, $D^K_m(G)$, is precisely $F_{m,0}(G)$. Later, in [PS2], Passi and Sehgal gave a similar result for the Lie dimension subgroups of G over K; namely, $D^K_{(m)}(G) = F_{m,1}(G)$.

A natural question arises: Can the subgroups $E^K_{m,k}(G)$ for $k \geq 2$ be described in a similar manner? One of the purposes of this chapter is to give an answer to this question. Indeed, in section 3.3 it is proven that the theorems of Lazard and Passi-Sehgal are examples of a more general phenomenon; that is, $E^K_{m+k,k}(G) = F_{m,k}(G)$ for all m and k. In the process we will give a unified proof for the aforemention results in [L] and [PS2].

Recently, in [Sh1], Shalev has demonstrated a nontrivial connection between these subgroups when K is a field of positive characteristic, which he has subsequently exploited in several interesting new papers (see References). The connection is that $(G, D^K_m(G)) = D^K_{(m+1)}(G)$, and in fact more generally that $(G, F_{m,k}(G)) = F_{m,k+1}(G)$ for all $m \geq 1$ and $k \geq 0$. Our alternate description of the $F_{m,k}(G)$ will enable us to somewhat broaden this result.

In the remaining sections 3.4-3.10, we show that much of the theory developed for ordinary and Lie dimension subgroups can be extended to include our k-dimension subgroups. Some applications are also made to the existing theory.

Let us now make some remarks regarding notation. As in Chapter Two, square brackets will indicate Lie commutators, while round brackets will denote group commutators. We will denote by $|i|_p$ the $p-$part of the integer i, whereas $|i|_{p'}$ will denote the part of i not divisible by p.

3.2 Basic lemmas.

Whenever R and G are implicitly understood from context, we will write $E_{m,k}$ for $E^R_{m,k}(G)$, and so forth.

The following lemma gives us some basic properties of the ideals $\Delta_{m,k}$.

3.2.1 LEMMA. *(i)* $\Delta_{m,k} = \Delta(G, G_m)$ *and* $E_{m,k} = G_m$ *whenever* $m \leq k+1$.

(ii) $\Delta_{m,k}\Delta_{n,k} \subseteq \Delta_{m+n-k,k}$ *for all* $m, n \geq 1$ *and* $k \geq 0$.

(iii) $\Delta_{m,k}{}^t \subseteq \Delta_{(m-k)t+k,k}$ *for all* $m, n, t \geq 1$ *and* $k \geq 0$.

Proof. (i) This is clear from the definition of $\Delta_{m,k}$.

(ii) Notice only that if $\sum(m_i - k) = m - k$ and $\sum(n_j - k) = n - k$, then $\sum(m_i - k) + \sum(n_j - k) = (m + n - k) - k$.

(iii) Use (ii) and induction on t.

□

For each $k \geq 0$ and $m \geq k+1$, set $I_{m,k} = \sum \prod_j \Delta(G_{m_j})$ where the sum is over all sequences $\{m_j\}$ for which $m \geq m_j \geq k+1$ and $\sum(m_j - k) = m - k$. Let $J_{m,k}$ denote the ideal in KG_{k+1} generated by $I_{m,k}$. Notice that $I_{m,k}$ generates $\Delta_{m,k}$ as an ideal in RG.

3.2.2 LEMMA. *(i)* $J_{m,k} = RG_{k+1} \cap \Delta_{m,k}$ *whenever* $m \geq k+1$.

(ii) $[J_{m,k}, \Delta(G_n)] \subseteq \Delta_{m+n,k}$ *if* $m \geq k+1$ *and* $n \geq 1$.

(iii) $[J_{m,k}, J_{n,k}] \subseteq J_{m+n,k}$ *whenever* $m, n \geq k+1$.

Proof. (i) It is trivial that $J_{m,k} \subseteq RG_{k+1} \cap \Delta_{m,k}$. Conversely suppose that $\delta \in RG_{k+1} \cap \Delta_{m,k}$. Then $\delta = \sum \prod_j \delta_{m_j} g$, where $g \in G$, $\delta_{m_j} \in \Delta(G_{m_j})$, $m \geq m_j \geq k+1$ and $\sum(m_j - k) = m - k$. Pick a transversal T of G_{k+1} in G and write every element $a \in G$ as $a = bt$ where $b \in G_{k+1}$ and $t \in T$. Then we have a K−homomorphism $KG \to KG_{k+1}$ induced by $a \mapsto b$. This map applied to the equation $\delta = \sum \prod_j \delta_{m_j} g$ shows that we may choose $g \in G_{k+1}$. Hence $\delta \in J_{m,k}$, so that $J_{m,k} = RG_{k+1} \cap \Delta_{m,k}$.

(ii) First let us show that $[I_{m,k}, \Delta(G_n)] \subseteq \Delta_{m+n,k}$ by induction on $m \geq k+1$. Suppose $m = k+1$. Using the identity $(x, y) - 1 = [x, y]y^{-1}x^{-1}$, we have

$$[I_{k+1,k}, \Delta(G_n)] = [\Delta(G_{k+1}), \Delta(G_n)] \subseteq \Delta(G, G_{k+1+n}) \subseteq \Delta_{k+1+n,k}.$$

Now suppose that $m > k+1$ and that $[I_{m',k}, \Delta(G_n)] \subseteq \Delta_{m'+n,k}$ whenever $m > m' \geq k+1$. Consider $\prod_{j=1}^{r} \Delta(G_{m_j})$ where $\sum_{j=1}^{r}(m_j - k)$, $m \geq m_j \geq k+1$ and without loss $r \geq 2$. By linearity it suffices to prove $[\prod_{j=1}^{r} \Delta(G_{m_j}), \Delta(G_n)] \subseteq \Delta_{m+n,k}$. Indeed with the use of Lemma 3.2.1(ii), the induction hypothesis and the identity $[ab,c] = a[b,c] + [a,c]b$ we have

$$\begin{aligned}[\prod_{j=1}^{r} \Delta(G_{m_j}), \Delta(G_n)] &\subseteq \prod_{j=1}^{r-1} \Delta(G_{m_j})[\Delta(G_{m_r}), \Delta(G_n)] + [\prod_{j=1}^{r-1} \Delta(G_{m_j}), \Delta(G_n)]\Delta(G_{m_r}) \\ &\subseteq \Delta_{m-(m_r-k),k}\Delta_{m_r+n,k} + [I_{m-(m_r-k),k}, \Delta(G_n)]\Delta_{m_r,k} \\ &\subseteq \Delta_{m+n,k}.\end{aligned}$$

Now observe that the identity $xy = x + x(y-1)$ with $x \in I_{m,k}$ and $y \in RG_{k+1}$ shows us that $J_{m,k} = I_{m,k} + I_{m+1,k}$. Hence $[J_{m,k}, \Delta(G_n)] \subseteq \Delta_{m+n,k}$ as required.

(iii) We claim $[I_{m,k}, I_{n,k}] \subseteq \Delta_{m+n,k}$ if $m, n \geq k+1$ by induction on $n \geq k+1$. If $n = k+1$, then $I_{k+1,k} = \Delta(G_{k+1})$, and so the basic step follows from part (ii). Consider $\prod_{j=1}^{s} \Delta(G_{n_j})$, where $n \geq n_j \geq k+1$, $\sum_{j=1}^{s}(n-k) = n-k$ and without loss $s \geq 2$. We need only show $[I_{m,k}, \prod_{j=1}^{s} \Delta(G_{n_j})] \subseteq \Delta_{m+n,k}$. But as in (i), we have

$$\begin{aligned}[I_{m,k}, \prod_{j=1}^{s} \Delta(G_{n_j})] &\subseteq \prod_{j=1}^{s} \Delta(G_{n_j})[I_{m,k}, \Delta(G_{n_s})] + [I_{m,k}, \prod_{j=1}^{s-1} \Delta(G_{n_j})]\Delta(G_{n_s}) \\ &\subseteq \Delta_{n-(n_s-k),k}\Delta_{m+n_s,k} + \Delta_{m+n-(n_s-k),k}\Delta_{n_s,k} \\ &\subseteq \Delta_{m+n,k}.\end{aligned}$$

This completes the proof of claim. Now since $J_{m,k} = I_{m,k} + I_{m+1,k}$, we see at once that $[J_{m,k}, J_{n,k}] \subseteq \Delta_{m+n,k}$. Finally by (i) we have $[J_{m,k}, J_{n,k}] \subseteq J_{m+n,k}$.

□

3.2.3 PROPOSITION. *For a fixed $k \geq 0$, $\{E_{m,k}\}_{m \geq 1}$ is an N-series for G, that is to say, $E_{1,k} = G$ and $(E_{m,k}, E_{n,k}) \subseteq E_{m+n,k}$ for all $m, n \geq 1$.*

Proof. Let $x \in E_{m,k}$ and $y \in E_{n,k}$. We must show that $(x,y) \in E_{m+n,k}$.

If $m, n \leq k+1$, then $(x,y) \in (G_m, G_n) \subseteq G_{m+n} \subseteq E_{m+n,k}$ by Lemma 3.2.1(i). Now suppose $m > k+1$ while $n \leq k+1$. Then $x \in E_{m,k} \subseteq G_{k+1}$, so that $x - 1 \in RG_{k+1} \cap \Delta_{m,k} = J_{m,k}$ by Lemma 3.2.2(i). Thus by Lemma 3.2.2(ii), $(x,y) - 1 = x^{-1}y^{-1}[x,y] \in RG[J_{m,k}, \Delta(G_n)] \subseteq \Delta_{m+n,k}$, so that $(x,y) \in E_{m+n,k}$. Finally suppose that $m, n > k+1$. Then $(x,y) - 1 = x^{-1}y^{-1}[x,y] \in RG[J_{m,k}, J_{n,k}]$ $\subseteq \Delta_{m+n,k}$ by Lemma 3.2.2(i) and (iii), yielding $(x,y) \in E_{m+n,k}$. Therefore, in every case, $(E_{m,k}, E_{n,k}) \subseteq E_{m+n,k}$.

□

3.3 $E^K_{m,k}(G)$ when K is a field of positive characteristic.

In this section the coefficient ring will be a field K of positive characteristic p. As usual, K and G will be suppressed in much of the notation.

An N-series $G = F_1 \supseteq F_2 \supseteq \cdots \supseteq F_m \supseteq \cdots$ is an N_p-series relative to a prime p if it has the additional property that $F_m^p \subseteq F_{mp}$. Associated to any N-series of G is a canonical filtration $\{A_m\}_{m \geq 1}$ of KG defined in the following manner. For each $g \in G$ define the height of g, $\nu(g)$ to be largest subscript m for which $g \in F_m$, if it exists, and $\nu(g) = \infty$ otherwise. Now for each integer m, let $A_m \subseteq KG$ be the K-linear span of all products of the form

$$(g_1 - 1)(g_2 - 1) \cdots (g_i - 1)$$

for some i with $\nu(g_1) + \nu(g_2) + \cdots + \nu(g_i) \geq m$. It is not difficult to see that A_m is an ideal of KG with the property that $A_m A_n \subseteq A_{m+n}$.

The first main result of this section is as follows.

3.3.1 THEOREM. *(i) For every group G and field K of characteristic $p > 0$, $F_{m,k}(G) = E^K_{m+k,k}(G)$ for each pair of integers $m \geq 1$ and $k \geq 0$.*

(ii) For a fixed $k \geq 0$, $\{F_{m,k}(G)\}_{m\geq 1}$ *is a restricted* N_p*-series for* G_{k+1}.

(iii) The canonical filtration associated to $\{F_{m,k}(G)\}_{m\geq 1}$ *is* $\{A_{m,k}\}_{m\geq 1}$, *defined by* $A_{m,k} = KG_{k+1} \cap \Delta^K_{m+k,k}(G)$.

3.3.2 LEMMA. *For each* $m \geq 1$ *and* $k \geq 0$, $F_{m,k} \subseteq E_{m+k,k}$.

Proof. Suppose i and j are such that $(i-k)p^j \geq m$. Then if $x \in G_i$, $x^{p^j} - 1 = (x-1)^{p^j} \in \Delta(G_i)^{p^j} \subseteq \Delta^{p^j}_{i,k} \subseteq \Delta_{(i-k)p^j+k,k} \subseteq \Delta_{m+k,k}$ by Lemma 3.2.1(iii). The lemma now follows.

□

The following is a special case of Dark's theorem [P; I.1.18].

3.3.3 LEMMA. *Suppose* $\phi : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N} \cup \{\infty\}$ *satisfies*

$$\phi(ri+sj, \binom{\alpha}{r}\binom{\beta}{s}) \geq \phi(i,\alpha) + \phi(i,\beta) \text{ and } \phi(i,p\alpha) \geq p\phi(i,\alpha),$$

for all α, β, r, s *such that* $1 \leq r \leq \alpha$ *and* $1 \leq s \leq \beta$. *Then* $\{F_m\}_{m\geq 1}$ *is an* N_p*-series of* F_1, *where*

$$F_m = \prod_{\phi(i,\alpha)\geq m} G_i^{\alpha}.$$

3.3.4 LEMMA. $\{F_{m,k}\}_{m\geq 1}$ *is a restricted* N_p*-series of* G_{k+1} *with canonical filtration* $\{A_{m,k}\}_{m\geq 1}$ *of* KG_{k+1} *defined by* $A_{m,k} = KG_{k+1} \cap \Delta_{m+k,k}$.

Proof. Let $\phi : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N} \cup \{\infty\}$ be defined by $\phi(i,\alpha) = (i-k)|\alpha|_p$ if $\alpha \in \mathbb{N}$ and $\phi(i,0) = \infty$. Using Lemma 3.3.3 applied to ϕ yields easily that $\{F_{m,k}\}_{m\geq 1}$ is a restricted N_p-series of G_{k+1}.

Now if $\sum_{j=1}^r m_j \geq m$, choose arbitrary $d_j \in F_{m_j,k}$, $1 \leq j \leq r$. Since Lemma 3.3.2 tells us that $d_j \in F_{m_j,k} \subseteq E_{m_j+k,k} \subseteq 1 + \Delta_{m_j+k,k}$, we have $\prod_{j=1}^r (d_j - 1) \in \prod_{j=1}^r \Delta_{m_j+k,k} \subseteq \Delta_{\left(\sum_{j=1}^r (m_j+k)\right)-(r-1)k,k} \subseteq \Delta_{m+k,k}$. Thus $A_m \subseteq KG_{k+1} \cap \Delta_{m+k,k}$.

Conversely, if $\delta \in KG_{k+1} \cap \Delta_{m+k,k}$, then $\delta \in J_{m+k,k}$ by Lemma 3.2.2(i). Hence $\delta \in A_{m,k}$ because $J_{m+k,k} = (\sum \prod_j \Delta(G_{m_j}))KG_{k+1} \subseteq (\sum \prod_j \Delta(F_{m_j-k,k}))KG_{k+1}$ $\subseteq A_{m,k}$ since here $\sum(m_j - k) = (m+k) - k = m$.

□

We need the following result which is implicit in the work of Jennings. It is proved in [P1; 3.3.6(iii)].

LEMMA 3.3.5. *If $\{A_m\}_{m\geq 1}$ is the canonical filtration of KG associated with an N_p-series $\{F_m\}_{m\geq 1}$ of G, then $F_m = G \cap (1 + A_m)$.*

Finally we are in a position to prove the theorem.

Proof of 3.3.1. Since $F_{m,k} \subseteq E_{m+k,k}$ from Lemma 3.3.2, we may assume $F_{m,k} = 1$ and show that this implies $E_{m+k,k} = 1$.

Indeed, suppose that $E_{m+k,k}(G/F_{m,k}) = 1$ and let $^- : KG \to K(G/F_{m,k})$ denote the canonical epimorphism. Now if $x - 1 \in \sum \prod_j \Delta(G, G_{m_j})$, then $\overline{x} - 1$ $\in \sum \prod_j \Delta(\overline{G}, \overline{G_{m_j}}) = \sum \prod_j \Delta(\overline{G}, (\overline{G})_{m_j}) = \Delta_{m+k,k}(\overline{G})$. Therefore $\overline{x} = 1$, so that $x \in F_{m,k}$ and $E_{m+k,k} \subseteq F_{m,k}$.

Clearly we may also assume G is finitely generated, and hence that G_{k+1} is a finite $p-$group. Thus by 3.3.4 and 3.3.5, we have

$$1 = F_{m,k} = G_{k+1} \cap (1 + A_{m,k}) = G_{k+1} \cap (1 + \Delta_{m+k,k}) = E_{m+k,k}$$

as required.

□

3.3.6 REMARKS. *(i) Shalev's theorem [Sh1; 1.2] now becomes $(G, E_{m,k}) = E_{m+1,k+1}$ for all $m \geq 1$ and $k \geq 0$.*

(ii) By setting $k = 0$ and $k = 1$ we retrieve the theorems of Lazard and of Passi and Sehgal; in particular,

$$D_m(G) = \prod_{ip^j \geq m} G_i^{p^j} \quad \text{and} \quad D_{(m)}(G) = \prod_{(i-1)p^j \geq m-1} G_i^{p^j}.$$

We are now in a position to extend Shalev's result. To do this, we will make use of the following technical lemma.

3.3.7 LEMMA. *One has* $[\Delta(G_{k+1})J_{m,k+1}, \Delta(G_n)] \subseteq J_{m+n,k+1}KG_{k+1}$, *whenever* $m \geq k+2$ *and* $n \geq k+1$.

Proof. The identity $[ab, c] = a[b, c] + [a, c]b$ together with Lemma 3.2.2(ii) informs us that

$$\begin{aligned}[\Delta(G_{k+1})J_{m,k+1}, \Delta(G_n)] &\subseteq \Delta(G_{k+1})[J_{m,k+1}, \Delta(G_n)] + [\Delta(G_{k+1}), \Delta(G_n)]J_{m,k+1} \\ &\subseteq \Delta(G_{k+1})J_{m+n,k+1} + \Delta(G_{k+1}, G_{n+k+1})J_{m,k+1} \\ &\subseteq J_{m+n,k+1}KG_{k+1}.\end{aligned}$$

This proves the lemma. □

Next we recall the following well known result, a proof of which may be found in [S; V.4.3].

3.3.8 LEMMA. *Let R be a ring of prime characteristic p. Then for $x, y \in R$ we have $[x, y, \ldots, y] = [x, y^{p^m}]$, where y appears in the first expression p^m times.*

3.3.9 THEOREM. *For all $m, n \geq 1$ and $k \geq 0$, we have $(E_{m,k}, E_{n,k}) \subseteq E_{m+n,k+1}$. In particular, then $(D_m, D_n) \subseteq D_{(m+n)}$.*

Proof. If $m, n \leq k+1$, then Lemma 3.2.1(i) gives us

$$(E_{m,k}, E_{n,k}) = (G_m, G_n) \subseteq G_{m+n} \subseteq E_{m+n,k+1}.$$

Next suppose that $m > k+1$ and $n \leq k+1$. Then repeated use of Remark 3.3.6(i) gives us

$$(E_{m,k}, E_{n,k}) = (E_{m,k}, G_n) \subseteq (E_{m,k}, G, \ldots, G) \subseteq E_{m+n,k+n} \subseteq E_{m+n,k+1}.$$

Finally we may assume that $m, n > k+1$. The group commutator identities $(xy,z) = (y,z)^x(x,z)$ and $(x,yz) = (x,y)(x,z)^y$ together with Theorem 3.3.1 enables us to reduce the problem to proving that $(g_i^{p^j}, h_r^{p^s}) \in E_{m+n,k+1}$, for all $g_i \in G_i$ and $h_r \in G_n$ whenever $(i-k)p^j \geq m-k$ and $(r-k)p^s \geq n-k$. Since $(g_i^{p^j}, h_r^{p^s}) \in E_{m+n,k+1}$ is equivalent to

$$[g_i^{p^j}, h_r^{p^s}] = [(g_i^{p^j}, h_r^{p^s}) - 1]h_r^{p^s} g_i^{p^j} \in \Delta_{m+n,k+1},$$

let us prove the latter instead. Two applications of Lemma 3.3.8 yields

$$\begin{aligned}
[g_i^{p^j}, h_r^{p^s}] &= [g_i^{p^j}, h_r, \ldots, h_r] \\
&= -[h_r, g_i^{p^j}, h_r, \ldots, h_r] \\
&= -[h_r, g_i, \ldots, g_i, h_r, \ldots, h_r] \\
&\in [\Delta(G_r), \Delta(G_i), \ldots, \Delta(G_i), \Delta(G_r), \ldots, \Delta(G_r)].
\end{aligned}$$

But since we can assume $r, i \geq k+1$, we have

$$\begin{aligned}
[\Delta(G_r), \Delta(G_i)] &\subseteq \Delta(G_{k+1}, G_{r+i}) \\
&\subseteq \Delta(G_{r+i})KG_{k+1} \\
&\subseteq J_{r+i,k+1}KG_{k+1}.
\end{aligned}$$

Observe that $i + r \geq 2(k+1) \geq k+2$. Now repeated applications of Lemma 3.3.7 tell us that $[g_i^{p^j}, h_r^{p^s}] \in J_{ip^j+rp^s,k+1} \subseteq J_{m+n,k+1}$ since

$$ip^j + rp^s \geq (m-k-kp^j) + (n-k+kp^s) = m+n+k(p^j+p^s-2) \geq m+n.$$

This proves the theorem.

□

3.4 $E^K_{m,k}(G)$ when K is a field of characteristic zero.

In light of Lemma 3.2.1, the proof of the following proposition follows in an analogous fashion to the Lie dimension subgroup case given in [PS2]. Despite this fact, we include a proof here for the sake of completeness.

3.4.1 THEOREM. *If K is a field of characteristic zero, then $E^K_{m,k}(G) = G_m$ when $m \leq k+1$ and $E^K_{m,k}(G) = G_{k+1} \cap \sqrt{G_m}$ when $m > k+1$. Here $\sqrt{G_m}$ is the isolator of G_m in G.*

Proof. We may assume $m > k+1$ by Lemma 3.2.1. Jennings [P; IV.1.5] proved the $k = 0$ case, that is, $E^K_{m,0}(G) = \sqrt{G_m}$. Hence by Lemma 3.2.1, $E^K_{m,k}(G) \subseteq G_{k+1} \cap \sqrt{G_m}$.

On the other hand, if $x \in G_{k+1} \cap \sqrt{G_m}$ for some $m > k+1$, then there is some r such that $x^r \in G_m$. Now consider the identity

$$x^r - 1 = \binom{r}{1}(x-1) + \binom{r}{2}(x-1)^2 + \cdots + \binom{r}{r}(x-1)^r. \tag{3.4.2}$$

Suppose, to the contrary, that $x - 1 \in \Delta^K_{j,k}(G)$ but $x - 1 \notin \Delta^K_{j+1,k}(G)$, for some j such that $k+1 \leq j < m$. Then together equation 3.4.2 and Lemma 3.2.1 imply that $x - 1 \in \Delta^K_{s,k}(G)$, where $s = min\{m, 2j-k\}$. However then $s \geq j+1$, a contradiction. Therefore, in fact $x - 1 \in \Delta^K_{m,k}(G)$, so that $x \in E^K_{m,k}(G)$.

□

Since when K is a field of positive characteristic we have $(G, E^K_{m,k}(G)) = E^K_{m+1,k+1}(G)$, one might wonder whether or not this holds true for other coefficient rings K as well. We shall see that this is not always the case. However, we do offer the following.

3.4.3 PROPOSITION. *If K is a field of characteristic zero, then $(G, E^K_{m,k}(G)) \subseteq E^K_{m+1,k+1}(G)$ for all $m \geq 1$ and $k \geq 0$.*

Proof. If $m \leq k+1$, then $E^K_{m,k}(G) = G_m$ by Theorem 3.4.1. Hence we have $(G, E^K_{m,k}(G)) = G_{m+1} \subseteq E^K_{m+1,k+1}(G)$, in this case. If $m > k+1$, then $E^K_{m,k}(G) \subseteq G_{k+1}$, so that $(G, E^K_{m,k}(G)) \subseteq G_{k+2}$. Also, since Proposition 3.2.3 holds for any coefficient ring, $(G, E^K_{m,k}(G)) = (E^K_{1,k}, E^K_{m,k}(G)) \subseteq E^K_{m+1,k} = G_{k+1} \cap \sqrt{G_{m+1}}$ by Theorem 3.4.1. Thus $(G, E^K_{m,k}(G)) \subseteq G_{k+2} \cap \sqrt{G_{m+1}} = E^K_{m+1,k+1}(G)$ in this case, too.

□

To illustrate that equality need not hold in Proposition 3.4.3, let us consider the following simple example of a group G where $(G, D_3(G)) = 1$ but $D_{(4)}(G) \neq 1$; that is, $(G, E^{\mathbf{Q}}_{3,0}) \subsetneq E^{\mathbf{Q}}_{4,1}$. Let G be the multiplicative group of matrices of the form

$$\begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix},$$

where $r, s \in \mathbf{Z}$ and $t \in \mathbf{Z}/2\mathbf{Z}$. It is easy to check that

$$G' = T(G) = \zeta(G) = \begin{pmatrix} 1 & 0 & \mathbf{Z}/2\mathbf{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the three subgroups are the derived, torsion and central subgroups of G, respectively. Thus $(G, E^{\mathbf{Q}}_{3,0}) = (G, \sqrt{G_3}) = (G, T(G)) = (G, \zeta(G)) = 1$, while $E^{\mathbf{Q}}_{4,1}$ $= G' \cap \sqrt{G_4} = G' \cap T(G) = G' \neq 1$, as claimed.

With Proposition 3.4.3 in hand, we get the following result, which is analogous to Theorem 3.3.9 and improves upon Proposition 3.2.3.

3.4.4 PROPOSITION. *Let K be a field of characteristic zero and suppose $m, n \geq 1$ and $k \geq 0$. Set $l = min\{n, k+1\}$. Then $(E^K_{m,k}(G), E^K_{n,k}(G)) \subseteq E^K_{m+n,l+k}(G)$.*

Proof. Certainly $E^K_{n,k}(G)$ is contained in G_l. Thus by iterative use of Proposition 3.4.3, we have

$$(E^K_{m,k}(G), E^K_{n,k}(G)) \subseteq (E^K_{m,k}(G), G_{k+1}) \subseteq (E^K_{m,k}(G), G, \ldots, G) \subseteq E^K_{m+l,k+l}(G),$$

so that by Theorem 3.4.1, $(E^K_{m,k}(G), E^K_{n,k}(G)) \subseteq G_{k+l} \cap \sqrt{G_{m+l}}$. On the other hand, by Proposition 3.2.3, we have

$$(E^K_{m,k}(G), E^K_{n,k}(G)) \subseteq E^K_{m+n,k}(G) = G_{k+1} \cap \sqrt{G_{m+n}}.$$

Therefore $(E^K_{m,k}(G), E^K_{n,k}(G)) \subseteq G_{k+l} \cap \sqrt{G_{m+n}} = E^K_{m+n,k+l}(G)$.

□

3.5 On the dimension subgroup conjecture.

The question of whether or not for all groups G and $m \geq 1$ we have $D^{\mathbb{Z}}_m(G) = G_m$, the so-called dimension subgroup conjecture, was open for more than thirty years. Then surprisingly, in 1972, Rips [R] constructed a finite 2-group with $D^{\mathbb{Z}}_4(G) \neq G_4$. This example was subsequently generalized by Gupta [G2] to show that the conjecture failed in general for all $n \geq 4$. Recently even the weaker conjectures that $D^{\mathbb{Z}}_{(m)}(G) = G_m$ and $D^{\mathbb{Z}}_{[m]}(G) = G_m$ were refuted by Hurley and Sehgal [HS] for all $m \geq 9$ and $m \geq 14$ respectively. Using their construction, Gupta and Srivastava in [GS] subsequently showed that in fact both of these alternate conjectures fail when $m \geq 9$. These Lie dimension subgroup conjectures remain open for $m = 7$ and 8, in light of the fact that Sandling [Sa1] has proven that $D^{\mathbb{Z}}_{(m)}(G) = G_m$ for all $m \leq 6$.

Because of the failure of the dimension and Lie dimension subgroup conjectures, perhaps our generalized dimension subgroups may be of some interest since they, in some sense, bridge the gap between $D^{\mathbb{Z}}_m(G)$ and G_m.

For brevity in what follows, $\mathbb{Z}$ and G sometimes may be suppressed.

Let us recall a crucial theorem from Sandling's paper [Sa1].

3.5.1 LEMMA. *For an arbitrary group H with centre $\zeta(H)$,*

$$H_3 = H \cap (1 + \Delta^3(H) + \Delta(H)\Delta(\zeta(H))).$$

We can exploit this result as did Sandling to obtain the following theorem.

3.5.2 THEOREM. *$E^{\mathbb{Z}}_{m,k}(G) = G_m$ in each of the following cases:*

(i) $G'_{k+1} = 1$, where G'_{k+1} is the derived subgroup of G_{k+1};

(ii) $m \leq 3k+3$ and G is arbitrary;

(iii) G is a p-group and the nilpotence class of G_{k+1} is less than p; and

(iv) $G^p_{k+1} = 1$ for some prime p.

Proof. (i) Suppose $G'_{k+1} = 1$. Since $G_m \subseteq E_{m,k}$, to show $E_{m,k}(G) = G_m$, we may assume $G_m = 1$ and prove $E_{m,k} = 1$. See the proof of 3.3.1 for a similar argument. By Lemma 3.2.1(i) we can assume that $m > k+1$, so that $E_{m,k} = G_{k+1} \cap (1 + J_{m,k})$ by Lemma 3.2.2(i). But since $G_m = 1$, certainly $J_{m,k} \subseteq \Delta^2(G_{k+1})$. Consequently, using the well known fact that $D^{\mathbb{Z}}_2(H) = H'$, we have

$$E_{m,k} \subseteq G_{k+1} \cap (1 + \Delta^2(G_{k+1})) = D^{\mathbb{Z}}_2(G_{k+1}) = G'_{k+1} = 1.$$

(ii) As above assume that $G_m = 1$. We claim that $J_{m,k} \subseteq \Delta^3(G_{k+1}) + \Delta(G_{k+1})\Delta(\zeta(G_{k+1}))$ if $m \leq 3k+3$. This is tantamount to showing that whenever $\sum_{j=1}^{r}(m_j - k) = m - k$, $\prod_{j=1}^{r} \Delta(G_{m_j})KG_{k+1} \subseteq \Delta^3(G_{k+1}) + \Delta(G_{k+1})\Delta(\zeta(G_{k+1}))$. But if $r = 1$ or $r \geq 3$, this is quite clear; thus suppose $r = 2$ and $m_1 \geq m_2$. It suffices to show that $m \geq m_1 - k - 1$, so that $G_{m_1} \subseteq \zeta(G_{k+1})$ since in this case $(G_{m_1}, G_{k+1}) \subseteq G_{m_1+k+1} \subseteq G_m = 1$. Suppose then, to the contrary, that $m \leq m_1 - k - 2$. Then we have $m - k = m_1 - k + m_2 - k \leq 2[(m-k-2)-k] = 2m - 3k + 4$; that is, $m \geq 3k+4$, a contradiction. Thus the claim is true.

Now by applying Lemma 3.5.1 to $H = G_{k+1}$, we have

$$\begin{aligned} E_{m,k} &= G_{k+1} \cap (1 + J_{m,k}) \\ &\subseteq G_{k+1} \cap (1 + \Delta^3(G_{k+1}) + \Delta(G_{k+1})\Delta(\zeta(G_{k+1}))) \\ &\subseteq (G_{k+1})_3 \\ &\subseteq G_{3k+3} \\ &= 1. \end{aligned}$$

This proves (ii).

(iii) This is similar to the proof of [Sa1; 3.3].

(iv) Since $E^{\mathbb{Z}/p\mathbb{Z}}_{m,k}(G) = G \cap (1 + \Delta^{\mathbb{Z}}_{m,k}(G) + p\mathbb{Z}G)$, then $E^{\mathbb{Z}}_{m,k}(G) \subseteq E^{\mathbb{Z}/p\mathbb{Z}}_{m,k}(G) = G_m$ by Theorem 3.3.1. Hence $E^{\mathbb{Z}}_{m,k}(G) = G_m$.

□

3.5.3 REMARKS. *(i) While it is presently unknown whether or not* $D^{\mathbb{Z}}_{(7)}(G) = E^{\mathbb{Z}}_{7,1}(G) = G_7$ *in general, it is true that at least* $E^{\mathbb{Z}}_{7,2}(G) = G_7$.

(ii) It is easy to see by Lemma 2.5.2 that for an arbitrary coefficient ring R *that* $\Delta^R_{m,2}(G) \subseteq \Delta^{[m]}_R(G)$. *Hence, in particular,*

$$E^{\mathbb{Z}}_{m,2}(G) \subseteq D^{\mathbb{Z}}_{[m]}(G) \subseteq D^{\mathbb{Z}}_{(m)}(G).$$

However, by our theorem above and the result of Gupta and Srivastava in [GS], there exists a group G *with* $E^{\mathbb{Z}}_{9,2}(G) = G_9 \neq D^{\mathbb{Z}}_{[9]}(G)$.

3.6 k-Nilpotence of the augmentation ideal.

For a fixed $k \geq 0$ consider the sequence

$$RG := \Delta_{0,k} \supseteq \Delta_{1,k} \supseteq \Delta_{2,k} \supseteq \cdots$$

If $\Delta_{t,k} = 0$ but $\Delta_{t-1,k} \neq 0$ for some integer t, we will say RG is *k-nilpotent* of *index* $t_k = t$.

3.6.1 PROPOSITION. *Let R be any commutative ring with identity. RG is k-nilpotent of index $m > k+1$ if and only if G is nilpotent, G_{k+1} is a finite $p-$group and p is nilpotent in R.*

Proof. We will use the well known special case of when $k = 0$, which is a theorem of Coleman. For its proof, see for example [S; I.2.21].

Suppose $\Delta_{m,k} = 0$. Then $E_{m,k} = 1$, so that G is nilpotent. Since $m > k+1$,

$$\Delta(G_{k+1})^{m-k} \subseteq \Delta_{k+1,k}^{m-k} \subseteq \Delta_{m,k} = 0$$

by Lemma 3.2.1(i) and (iii). Thus we have that G_{k+1} is a finite $p-$group and p is nilpotent in R.

Conversely, suppose that G is nilpotent and G_{k+1} is a finite p-group. By the $k = 0$ case, we can find an integer t such that $\Delta(G_{k+1})^t = 0$ and $G_t = 1$. Set $m = t^2 + k$. We claim that $\Delta_{m,k} = 0$.

If $\sum_{j=1}^{r}(m_j - k) = m - k$ with all $m_j \geq k+1$, then $r \geq t$ or $m_j \geq t$ for some j. Indeed otherwise

$$t^2 = m - k = \sum_{j=1}^{r}(m_j - k) < \sum_{j=1}^{r}(t-k) < t(t-k) \leq t^2,$$

an impossibility. Hence $\Delta_{m,k} = \sum \prod_j \Delta(G, G_{m_j}) = 0$, as claimed.

□

Thus we have generalized Coleman's classification of group rings with nilpotent augmentation ideal. In a similar fashion in section 3.7, we will obtain results anal ogous to [Ha], [PrPS] and [MW], which classify when KG is *residually k-nilpotent*, that is to say, $\bigcap_{m\geq 1} \Delta_{m,k} = 0$.

Let us now restrict our attention to when K is a field of characteristic $p > 0$ and G is a finite $p-$group. By Theorem 3.3.1 we can apply Lazard's method [P1; 3.3.6] to the filtration $\{A_{m,k}\}_{m\geq 0}$ of KG_{k+1}, defined by $A_{0,k} = KG_{k+1}$ and $A_{m,k} = KG_{k+1} \cap \Delta_{m+k,k}$ for $m \geq 1$, to describe $f_{i,k} := dim_K A_{i,k}/A_{i+1,k}$, $i \geq 0$, in terms of the subgroups $F_{m,k}$, $m \geq 1$. In fact, if we set $[F_{m,k} : F_{m+1,k}] = p^{d_{m,k}}$ for each $m \geq 1$, then the generating function $\sum_{i\geq 0} f_{i,k}X^i \in \mathbb{Z}[X]$ is given by

$$\sum_{i\geq 0} f_{i,k}X^i = \prod_{m\geq 1} \left(\frac{X^{mp}-1}{X^m-1}\right)^{d_{m,k}}. \tag{3.6.2}$$

Also $A_{t,k} = 0$ but $A_{t-1,k} \neq 0$, where $t = 1 + \sum_{m\geq 1}(p-1)md_{m,k}$.

Now set $c_{0,k} = dim_K KG/\Delta(G, G_{k+1})$ and $c_{i,k} = dim_K \Delta_{i+k,k}/\Delta_{i+k+1,k}$ for $i \geq 1$. We could call $\{c_{i,k}\}_{i\geq 0}$ the *k-Loewy series* of KG.

3.6.3 PROPOSITION. *(i) The generating function $\sum_{i\geq 0} c_{i,k}X^i$ is given by*

$$\sum_{i\geq 0} c_{i,k}X^i = [G : G_{k+1}] \prod_{m\geq 1} \left(\frac{X^{mp}-1}{X^m-1}\right)^{d_{m,k}}.$$

(ii) The $k-$nilpotence index of KG is $t_k = (k+1) + \sum_{m\geq 1}(p-1)md_{m,k}$.

Proof. (i) We need to show $c_{i,k} = [G : G_{k+1}]f_{i,k}$. But this is clear since by Lemma 3.2.2 we have $\Delta_{i+k,k} = A_{i,k}KG$ for all $i \geq 1$.

(ii) This follows at once from the discussion above.

□

Shalev's work [Sh1; Section 1] is now readily generalized to show the sequence $\{d_{m,k}\}_{m\geq 1}$ is subject to the following arithmetical constraints.

3.6.4 PROPOSITION. *(i) If $d_{m,k} = 0$ and $l \geq m$ satisfies $|l|_{p'} \geq |m|_{p'}$, then $d_{l,k} = 0$.*

(ii) If $d_{m,k} = 0$ and $exp(G_{k+1})|pm$ then $d_{l,k} = 0$ for all $l \geq m$.

(iii) If $d_{m,k} = 0$ and $l \geq m/p$, then $d_{p',k} \leq d_{l,k}$.

Now with Proposition 3.6.4(ii) playing the part of the main lemma in [Sh2, Theorem B], we can obtain the following extension of his results.

3.6.5 PROPOSITION. *(i) $t_k \leq |G_{k+1}| + k$.*

(ii) Suppose p is odd, and let i be a fixed non-negative integer. If G_{k+1} is sufficiently large, then $t_k > |G_{k+1}|/p^i$ if and only if G_{k+1} possesses a cyclic subgroup of index p^i.

3.7 Residual k-nilpotence of the augmentation ideal.

Now let us turn to the question of residual k-nilpotence. For a prime p, let $\mathcal{K}_{p,k}$ be the class of nilpotent groups with G_{k+1} a $p-$group of bounded exponent and let $\mathcal{K}_k = \cup_p \mathcal{K}_{p,k}$. Set $\mathcal{K}_{0,k}$ to be the class of nilpotent groups G with G_{k+1} torsion free. The following is a generalization of results due to Hartley [Ha] and Parmenter, Passi and Sehgal [PrPS], both of which appear in [P; VI].

3.7.1 PROPOSITION. *Let G be a group and K be a field of characteristic $p \geq 0$. The following statements are equivalent:*

(i) $\bigcap_{m \geq 1} \Delta^K_{m,k}(G) = 0$;

(ii) $\bigcap_{m \geq 1} E^K_{m,k}(G) = 1$; and

(iii) G is residually $\mathcal{K}_{p,k}$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial in view of Theorems 3.3.1 and 3.4.1. To prove (iii) $\Rightarrow$ (i), we will use the case $k = 0$ proven in [P; VI.2.26]. Suppose then that $k > 0$.

Let us prove first that $G \in \mathcal{K}_{p,k}$ implies that $\bigcap_{m\geq 1} \Delta_{m,k} = 0$. Since G is nilpotent, it is easy to see $\bigcap_{m\geq 1} \Delta_{m,k} \subseteq \bigcap_{m\geq 1}(\Delta^m(G_{k+1})KG) = \bigcap_{m\geq 1}(\Delta^m(G_{k+1}))KG$. But by the $k = 0$ case we have $\bigcap_{m\geq 1} \Delta^m(G_{k+1}) = 0$. Hence $\bigcap_{m\geq 1} \Delta_{m,k} = 0$.

Now let us consider the general case of G residually $\mathcal{K}_{p,k}$. Suppose to the contrary that there is some $x \in \bigcap_{m\geq 1} \Delta_{m,k}$ and $x \neq 0$. Put $x = \sum \alpha_i g_i$, where $\alpha_i \in K$ and $g_i \in G$. For each i we can find $N_i \triangleleft G$ with $g \notin N_i$ and $G/N_i \in \mathcal{K}_{p,k}$. Let $N = \bigcap_i N_i$. Then $G/N \in \mathcal{K}_{p,k}$ since G/N can be embedded in $Dr_i(G/N_i) \in \mathcal{K}_{p,k}$. Now consider the canonical map $^- : KG \to K(G/N)$. By construction $\overline{x} \neq 0$; however $\overline{x} \in \bigcap_{m\geq 1} \Delta_{m,k}(\overline{G})$, contradicting the paragraph above. This proves the proposition.

□

Recall that G is discriminated by a class $\mathcal{X}$ if for any finite subset $g_1, g_2, \ldots, g_n$ of distinct elements of G, there is some homomorphism $\phi : G \to H$ such that $H \in \mathcal{X}$ and the elements $\phi(g_1), \phi(g_2), \ldots, \phi(g_n)$ are still distinct. We now consider the integral case, which generalizes results of Lichtman [Li] and Musson and Weiss [MW].

3.7.2 LEMMA. *Let p be a prime and G be a group. Then $G/E_{n,k}^{\mathbb{Z}/p^m\mathbb{Z}}(G) \in \mathcal{K}_{p,k}$ for all $n \geq 1$.*

Proof. Let $n \geq 1$ be given. Since $G_n \subseteq E_{n,k}^{\mathbb{Z}/p^m\mathbb{Z}}(G)$, it suffices to show that $G_{k+1}^{p^N} \subseteq 1 + \Delta_{n,k}^{\mathbb{Z}/p^m\mathbb{Z}}(G)$, for some N. But by Lemma 3.2.1 this is so for $p^N \geq n - k$.

□

3.7.3 PROPOSITION. *Let G be a group. $\bigcap_{m\geq 1} \Delta_{m,k}^{\mathbb{Z}}(G) = 0$ if and only if either G is residually $\mathcal{K}_{0,k}$, or G is discriminated by the class $\mathcal{K}_k$.*

Proof. For the most part, we will follow the methods of [MW; Theorem A].

($\Leftarrow$) If G is $\mathcal{K}_{0,k}$, then by Proposition 3.7.1 we have $\Delta^{\mathbb{Z}}_{m,k}(G) \subseteq \Delta^{\mathbb{Q}}_{m,k}(G) = 0$. So assume that G is discriminated by $\mathcal{K}_k$ and $0 \neq \sum_i m_i(g_i - 1) \in \bigcap_{m\geq 1} \Delta^{\mathbb{Z}}_{m,k}(G)$, where $g_1, g_2, \ldots, g_n$ are distinct non-indentity elements of G. Then there is a homomorphism $\phi : G \to H$, where $H \in \mathcal{K}_{p,k}$ for some prime p and the $\phi(g_i)$ are distinct non-indentity elements of H. This implies that $0 \neq \sum_i m_i(\phi(g_i) - 1) \in \bigcap_{m\geq 1} \Delta^{\mathbb{Z}}_{m,k}(H)$, contradicting Proposition 3.7.1.

($\Rightarrow$) Case (i): Suppose that G_{k+1} is not torsion-free. Then G_{k+1} has an element g of prime order p. We claim that in this case, necessarily $\bigcap_{m,n} E^{\mathbb{Z}/p^m\mathbb{Z}}_{n,k}(G) = 1$. By Lemma 3.2.1 for each $n \geq 1$ we can find an $r = r(n)$ such that $(g-1)^r \in \Delta^{\mathbb{Z}}_{n,k}(G)$. Since the order g is p, it then follows by equation 3.4.2 that $p^{r-1}(g-1) \in \Delta^{\mathbb{Z}}_{n,k}(G)$. Let $h \in \bigcap_{m,n} E^{\mathbb{Z}/p^m\mathbb{Z}}_{n,k}(G)$. Then $h - 1 \in \Delta^{\mathbb{Z}}_{n,k}(G) + p^m\mathbb{Z}G$ for all m, n. Hence $(g-1)(h-1) \in \bigcap_{m\geq 1} \Delta^{\mathbb{Z}}_{m,k}(G) = 0$, and consequently $h = 1$. Now applying Lemma 3.7.2, we see that G is residually $\mathcal{K}_{p,k}$, and thus G is discriminated by $\mathcal{K}_k$ (see the proof of Proposition 3.7.1).

Case (ii): G_{k+1} is torsion-free. If $\bigcap_{m\geq 1} E^{\mathbb{Q}}_{m,k}(G) = 0$, then G is residually $\mathcal{K}_{0,k}$ by Proposition 3.7.1. Thus we may assume that there exists $1 \neq g_0 \in \bigcap_{m\geq 1} E^{\mathbb{Q}}_{m,k}(G)$ and that G is not discriminated by $\mathcal{K}_k$. Then for each n, there is an integer $l = l(n)$ with

$$l(g_0 - 1) \in \Delta^{\mathbb{Z}}_{n,k}(G). \tag{3.7.4}$$

Also, by Lemma 3.7.2, there exist finitely many non-indentity elements $g_1, g_2, \ldots, g_r$, say, in G_{k+1}, such that for every $m, n \geq 1$ and prime p, at least one of $g_1, g_2, \ldots, g_r$ belongs to $E^{\mathbb{Z}/p^m\mathbb{Z}}_{n,k}(G)$. Therefore we always have

$$(g_1 - 1)(g_2 - 1)\cdots(g_r - 1) \in \Delta^{\mathbb{Z}}_{n,k}(G) + p^m\mathbb{Z}G. \tag{3.7.5}$$

Now putting equations 3.7.4 and 3.7.5 together we obtain that

$$(g_0 - 1)(g_1 - 1)\cdots(g_r - 1) \in \bigcap_{n\geq 1} \Delta^{\mathbb{Z}}_{n,k}(G) = 0.$$

Since each g_i has infinite order, this is impossible. Indeed, let s be minimal such that $(g_0-1)(g_1-1)\cdots(g_s-1)=0$. Then $\tau=\sum_h \tau_h h := (g_0-1)(g_1-1)\cdots(g_{s-1}-1)\neq 0$, but $\tau(g_s^t-1)=0$ for all $t\in\mathbf{N}$. Replacing τ by $h^{-1}\tau$, where $\tau_h\neq 0$, if necessary, we can assume that $\tau_1\neq 0$. Now in $g_s^t\tau$, the coefficient of g_s^t is τ_1. It follows that $\tau_{g_s^t}=\tau_1$ for all $t\in\mathbf{N}$, contradicting the fact that g_s has infinite order. Thus G actually is discriminated by $\mathcal{K}_k$.

□

3.8 On the modular isomorphism problem.

Much effort has been spent in obtaining isomorphism invariants of $\mathbf{F}_pG$, where $\mathbf{F}_p$ is the field of p elements and G is a finite p-group. Some of the best results include Sandling's result [Sa2] that $G/G_2^pG_3$, in other words $G/D_{(3)}(G)$ when $p>2$, is an invariant, and the result of Ritter and Sehgal [RS] and Furukawa [F] that $D_n(G)/D_{2n+1}(G)$ is an invariant for each $n\geq 1$.

In this section, we make a modest extension of these results using the theory that has been developed for Lie dimension subgroups.

3.8.1 THEOREM. *If G is finite p-group, then $D_{(m)}(G)/D_{(m+1)}(G)$ is determined by the isomorphism class of $\mathbf{F}_pG$ for all $m\geq 1$.*

Proof. When $m=1$, we have $D_{(1)}(G)/D_{(2)}(G)=G/G'$, which was shown to be an invariant of $\mathbf{F}_pG$ by Ward in [W]. Thus we may assume that $m\geq 2$. For each $i\geq 0$ set $f_{i,1} := dim_{\mathbf{F}_p} J_{i+1,1}(G)/J_{i+2,1}(G)$, where $J_{1,1}(G)=\mathbf{F}_pG'$, and for each $m\geq 1$ put $[D_{(m+1)}(G) : D_{(m+2)}(G)] = p^{d_{m,1}}$. Then by the discussion preceding Proposition 3.6.2, we have that

$$\sum_{i\geq 0} f_{i,1}X^i = \prod_{n\geq 1}\left(\frac{X^{np}-1}{X^n-1}\right)^{d_{n,1}},$$

so that the $f_{i,1}$ are determined in terms of the $d_{n,1}$. More importantly for us the converse is also true: namely, the $d_{n,1}$ are determined in terms of the $f_{i,1}$. If we can show that the $f_{i,1}$ are isomorphism invariants of $\mathbf{F}_pG$, then the proposition will follow since $D_{(m)}(G)/D_{(m+1)}(G)$ is merely a finite elementary abelian p-group, and hence is determined up to isomorphism by its order. Indeed, since $m \geq 2$, by Theorem 3.3.1(ii), we have

$$(D_{(m)}(G))^p = (F_{m-1,1}(G))^p \subseteq F_{p(m-1),1}(G) = D_{(p(m-1)+1)}(G) \subseteq D_{(m+1)}(G),$$

and by Proposition 3.2.3, $(D_{(m)}(G))' \subseteq D_{(2m)}(G) \subseteq D_{(m+1)}(G)$. Therefore we need only prove the following lemma.

□

3.8.2 LEMMA. *$f_{0,1} = 1$ and $f_{i,1} = dim_{\mathbf{F}_p} \frac{\Delta^{(i+1)}(G)}{\Delta(G)\Delta^{(i+1)}(G)+\Delta^{(i+1)}(G)\Delta(G)}$ for all $i \geq 1$.*

Thus in particular, the $f_{i,1}$ are all invariants of $\mathbf{F}_pG$.

Proof. Consider the map $J_{i+1,1}(G) \to \frac{\Delta^{(i+1)}(G)}{\Delta(G)\Delta^{(i+1)}(G)+\Delta^{(i+1)}(G)\Delta(G)}$, induced by inclusion. Let N be the kernel of this map. Then clearly $J_{i+2,1}(G) \subseteq N$. Also, by using the same trick as in the proof of Lemma 3.2.2(i), we have

$$\begin{aligned} N &= J_{i+1,1}(G) \cap (\Delta(G)\Delta^{(i+1)}(G) + \Delta^{(i+1)}(G)\Delta(G)) \\ &\subseteq \mathbf{F}_pG' \cap (\Delta(G)\Delta^{(i+1)}(G) + \Delta^{(i+2)}(G)) \\ &= \mathbf{F}_pG' \cap (\Delta(G)J_{i+1,1}(G) + \Delta^{(i+2)}(G)) \\ &= \mathbf{F}_pG' \cap (\Delta(G')J_{i+1,1}(G) + J_{i+2,1}(G)) \\ &= J_{i+2,1}(G). \end{aligned}$$

Thus, as $\mathbf{F}_p$-vector spaces, $\frac{J_{i+1,1}(G)}{J_{i+2,1}(G)} \cong \frac{\Delta^{(i+1)}(G)}{\Delta(G)\Delta^{(i+1)}(G)+\Delta^{(i+1)}(G)\Delta(G)}$ for all $i \geq 1$. When $i = 0$, we have $f_{0,1} = dim_{\mathbf{F}_p} \mathbf{F}_pG'/\Delta G' = 1$.

□

3.8.3 COROLLARY. *If G is a finite p-group, then for all $m \geq 1$:*

(i) $D_{(m)}(G)/D_{(2m)}(G)$ is an invariant of $\mathbf{F}_pG$ if $p > 2$, and

(ii) $D_{(m)}(G)/D_{(2m-1)}(G)$ is an invariant of $\mathbf{F}_2G$.

Proof. (i) By Theorem 3.8.1, we can assume $m \geq 2$. Then, as in the proof of Theorem 3.8.1, $(D_{(m)}(G))^p \subseteq D_{(p(m-1)+1)}(G) \subseteq D_{(2m)}(G)$ and $(D_{(m)}(G))' \subseteq D_{(2m)}(G)$. Consequently $D_{(m)}(G)/D_{(2m)}(G)$ is an elementary abelian p-group, which is determined up to isomorphism by its order $[D_{(m)}(G) : D_{(2m)}(G)] = \prod_{i=0}^{m-1}[D_{(m+i)}(G) : D_{(m+i+1)}(G)]$. But of course this product is an invariant by Theorem 3.8.1.

(ii) When $p = 2$ we can only say $(D_{(m)}(G))^2 \subseteq D_{(2m-1)}(G)$. Now the proof proceeds as in part (i).

□

3.8.4 COROLLARY. *If G is a finite p-group and $\mathbf{F}_pG \cong \mathbf{F}_pH$ as $\mathbf{F}_p$-algebras, then $D_{(m)}(G) = 1$ if and only if $D_{(m)}(H) = 1$.*

Proof. Suppose $D_{(m)}(G) = 1$. Then by Theorem 3.8.1, we have $D_{(m)}(H) = D_{(m+i)}(H)$ for all $i \geq 0$. Since $D_{(m)}(G) = 1$, G is nilpotent. Now applying Proposition 3.7.1 we obtain $\bigcap_{m\geq 1} \Delta^{(m)}(G) = 0$, and consequently $\bigcap_{m\geq 1} \Delta^{(m)}(H) = 0$. It now follows that $D_{(m)}(H) = 1$. By symmetry we are done.

□

3.8.5 REMARKS. *(i) Of course, for each i such that $1 \leq i \leq m$, $D_{(m)}(G)/D_{(m+i)}(G)$ is also an invariant of $\mathbf{F}_pG$ if $p > 2$.*

(ii) Suppose $p > 2$. Then whether or not G' is a finite p-group is an isomorphism invariant of $\mathbf{F}_pG$ by [S; V.3.1]. All we really needed to prove Theorem 3.8.1 was

that G' is a finite p-group and that this was an invariant of $\mathbf{F}_pG$; consequently, all the results of this section hold equally well under the weaker hypothesis that only G' is a finite p-group.

3.9 The graded Lie algebra $\mathcal{L}_k^K(G)$ associated with $\{F_{m,k}(G)\}_{m\geq 1}$.

Let K be a field of characteristic $p > 0$, and G be any group. We know by Lemma 3.3.4 that $\{F_{m,k}(G)\}_{m\geq 1}$ is an N_p-series of G_{k+1}. Lazard [L] has described a graded, restricted Lie K-algebra structure which naturally arises from any N_p-series. Set

$$\mathcal{L}_k^K(G) = K \otimes_{\mathbf{F}_p} \left(\bigoplus\nolimits_{m\geq 1} F_{m,k}(G)/F_{m+1,k}(G) \right).$$

Then via the abelian group structure of $\bigoplus_{m\geq 1} F_{m,k}(G)/F_{m+1,k}(G)$, $\mathcal{L}_k^K(G)$ is a K-vector space. The grading is given by the convention that the elements of $F_{m,k}(G)/F_{m+1,k}(G)$ have homogeneous degree m. The Lie operation is defined in the following way. For homogeneous elements $\overline{x_m} \in F_{m,k}(G)/F_{m+1,k}(G)$ and $\overline{x_n} \in F_{n,k}(G)/F_{n+1,k}(G)$, set

$$[\overline{x_m}, \overline{x_n}] = \overline{(x_m, x_n)} \in F_{m+n,k}(G)/F_{m+n+1,k}(G),$$

where $\overline{x_i} = x_iF_{i+1,k}(G)$. Finally, define the restriction map by

$$\overline{x_m}^{[p]} = \overline{x_m^p} \in F_{mp,k}(G)/F_{mp+1,k}(G).$$

To see that this is all well-defined, refer to [P; VIII].

In this section we shall extend Quillen's theorem [Q] to our generalized dimension subgroups. We adopt the notation introduced in section 3.6. Put $gr_k(KG) = \bigoplus_{m\geq 0} A_{m,k}/A_{m+1,k}$. Then $gr_k(KG)$ can be considered as a graded K-algebra in the following manner. The abelian groups $A_{m,k}/A_{m+1,k}$ induce an addition on

$gr_k(KG)$; the elements of $A_{m,k}/A_{m+1,k}$ are considered to have homogeneous degree m; and, finally a K-algebra multiplication is defined by

$$\underline{x_m}\underline{x_n} = \underline{x_m x_n} \in A_{m+n,k}/A_{m+n+1,k},$$

where $\underline{x_i} = x_i + A_{i+1,k} \in A_{i,k}/A_{i+1,k}$. Notice that this multiplication is well-defined by Lemmas 3.2.1 and 3.2.2.

Let $u(L)$ denote the restricted universal enveloping algebra of a restricted Lie algebra L. For the definition of $u(L)$ and its basic properties, see, for example, [SF]. We are now ready to state our result.

3.9.1 THEOREM. *If G_{k+1} is a finite p-group and K is a field of characteristic $p > 0$, then*

$$u(\mathcal{L}_k^K(G)) \cong gr_k(KG)$$

as graded K-algebras, the grading on $u(\mathcal{L}_k^K(G))$ being the one induced by the grading of $\mathcal{L}_k^K(G)$.

Proof. The inclusion map $F_{m,k}(G)/F_{m+1,k}(G) \to A_{m,k}/A_{m+1,k}$ given by $\overline{x_m} \to \underline{x_m - 1}$ induces a graded, restricted Lie K-algebra monomorphism $\theta : \mathcal{L}_k^K(G) \to gr_k(KG)$. In turn, by the universal property of $u(\mathcal{L}_k^K(G))$, θ induces a homomorphism of graded K-algebras $\theta : u(\mathcal{L}_k^K(G)) \to gr_k(KG)$. The grading of $u(\mathcal{L}_k^K(G))$ is the one induced by $\mathcal{L}_k^K(G)$. This map θ is necessarily epic since clearly

$$\theta\left(\frac{F_{1,k}(G)}{F_{2,k}(G)}\right) = \frac{G_{k+1} - 1 + A_{2,k}}{A_{2,k}}$$

generates $gr_k(KG)$ as a K-algebra.

We need to show only that θ is monic. Let $x_{m,1}, x_{m,2}, \ldots, x_{m,d_{m,k}}$ be an $\mathbf{F}_p$-basis of $F_{m,k}(G)/F_{m+1,k}(G)$ for each $m \geq 1$. Order the $x_{i,j}$ lexicographically. Then by the Poincaré-Birkoff-Witt Theorem for restricted universal enveloping algebras

(see [SF]), a K-basis for $u(\mathcal{L}_k^K(G))_t$, the t-th homogeneous component of $u(\mathcal{L}_k^K(G))$, is given by all the (ordered) monomials $\prod x_{m,j}^{\alpha_{m,j}}$, where the product is over all $m \geq 1$ and $1 \leq j \leq d_{m,k}$ such that $0 \leq \alpha_{m,j} \leq p-1$ and only t of the $\alpha_{m,j}$ are nonzero. Examining the proof of equation 3.6.2 in [P1; 1.3.6(ii)] we now see immediately that $dim_K u(\mathcal{L}_k^K(G))_t = dim_K A_{t,k}/A_{t+1,k}$. This proves the theorem.

□

3.9.2 REMARKS. *(i) A similar result holds for fields K of characteristic zero and groups G with G_{k+1} finitely generated and torsion-free; namely, $gr_k(KG)$ is the (unrestricted) universal enveloping algebra of $\mathcal{L}_k^K(G)$.*

(ii) The hypotheses that G_{k+1} be a finite p-group when $p > 0$ and that G_{k+1} be finitely generated and torsion-free when $p = 0$ are not necessary. Both results hold for arbitrary groups G using a reduction argument similar to the one given for the $k = 0$ case given in [P; VIII].

(iii) We have $(F_{m,k}(G), F_{n,k}(G)) \subseteq F_{m+n+k,k}(G)$ by Theorem 3.3.1 and Proposition 3.2.3. Thus for $k > 0$, $\mathcal{L}_k^K(G)$ is a restricted abelian Lie algebra, so that $u(\mathcal{L}_k^K(G))$ is merely a commutative algebra.

Chapter Four

POLYNOMIAL IDENTITIES IN GRADED GROUP RINGS

4.1 Introduction.

Let L be a restricted Lie algebra over the field K of characteristic $p > 0$, and let $u(L)$ be its restricted universal enveloping algebra. Using techniques similar to those used in the study of group algebras satisfying polynomial identities, Passman [P2; Theorem 6.1] has recently given a precise answer to the problem of describing when $u(L)$ satisfies a polynomial identity. His description is as follows. Denote by $\omega(L)$, the augmentation ideal of $u(L)$, that is, the kernel of the natural epimorphism $u(L) \to u(L/L) = K$.

4.1.1 THEOREM. *Let L be a restricted Lie algebra. Then $u(L)$ satisfies a polynomial identity if and only if L has restricted subalgebras $A \supseteq B$ such that*

(i) $dim_K L/A < \infty$ and $dim_K B < \infty$,

(ii) B is a central restricted ideal of A with A/B abelian, and

(iii) $\omega(B)$ is nilpotent.

While this solves the problem in general, it does not, however, answer the problem for specific polynomial identities. It is easy to see that $u(L)$ is a PI–algebra if and only if $\omega(L)$ is a PI–algebra. The initial objective of this chapter is to determine precisely when $\omega(L)$ is, firstly, nilpotent as an ideal, secondly, strongly Lie nilpotent, and, lastly, Lie n–Engel, for some n. These three properties correspond to certain classes of polynomial identities.

Let R be any Lie algebra over K. Recall that if R is an associative algebra, then R can be viewed as a Lie algebra via the natural Lie bracket $[x, y] = xy - yx$, where $x, y \in R$. For each positive integer m, denote by $\gamma_m(R)$, the m-th term of the Lie lower central series of R, which is defined to be the K–subspace generated by all the

(left normed) Lie commutators of the form $[x_1, x_2, \ldots, x_m]$ $(x_i \in R)$. If $\gamma_m(R) = 0$ for some m, then we say that R is Lie nilpotent of class at most $m-1$. There is another natural way of defining Lie nilpotence of an associative algebra R that will be of interest to us. Define inductively the Lie powers of R by setting $R^{(1)} = R$ and $R^{(m+1)}$ to be (associative) ideal generated by elements of the form $[x, y]$, where $x \in R^{(m)}$ and $y \in R$. If $R^{(m)} = 0$ for some m, then we call R *strongly* Lie nilpotent (since clearly $\gamma_m(R) \subseteq R^{(m)}$). A Lie algebra R is Lie n–Engel if every $x, y \in R$ satisfies the identity $[x, {}_n y] = [x, y, y, \ldots y] = 0$, where y appears in the expression n times. Clearly, if R is Lie nilpotent, it is Lie n–Engel, for some n. Now suppose that R is a restricted Lie algebra with p–mapping $x \to x^{[p]}$. An element $x \in R$ is p–nilpotent if there is an nonnegative integer m such that $x^{[p]^m} = 0$; a subset S of R is p–nil if each element of S is p–nilpotent; and finally, S is p–nilpotent if there exists an m such that $S^{[p]^m} = \{x^{[p]^m} \mid x \in S\} = 0$.

Notice that $u(L)$ is strongly Lie nilpotent (Lie n–Engel) if and only if $\omega(L)$ is strongly Lie nilpotent (Lie n–Engel) since $u(L) = K \oplus \omega(L)$.

Our first result is as follows. Let $L' = \gamma_2(L)$ for any Lie algebra L.

4.1.2 THEOREM. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$.*

(i) $\omega(L)$ is nilpotent if and only if L is finite dimensional and p–nilpotent.

(ii) $u(L)$ is strongly Lie nilpotent if and only if L is Lie nilpotent and L' is finite dimensional and p–nilpotent.

We say that a restricted Lie algebra L admits a grading (actually an $\mathbb{N}$–grading) if L can be written as the direct sum of linear spaces L_n $(n \in \mathbb{N})$ satisfying $[L_m, L_n] \subseteq L_{m+n}$ and $L_m^{[p]} \subseteq L_{mp}$ for all $m, n \in \mathbb{N}$. Shalev [Sh3; 4] has pointed out that if L admits a grading, then the A and B in Passman's theorem can be taken to be

graded as well. Now, as an application of Theorem 4.1.2, we will see that we can rewrite Passman's theorem in the following simplified manner.

4.1.3 COROLLARY. *Let L be a (graded) restricted Lie algebra over a field of characteristic $p > 0$. Then $u(L)$ satisfies a polynomial identity if and only if L has a (graded) restricted subalgebra A such that*

(i) $dim_K L/A < \infty$,

(ii) A is Lie nilpotent of class at most two, and

(iii) A' is finite dimensional and p–nilpotent.

Furthermore, it will be shown that if L admits a grading, then in fact the A above can be chosen to be an ideal of L. It is quite conceivable that this is also true in the non–graded case. Theorem 4.1.2 can be strengthened as well if we assume that L admits a grading.

4.1.4 THEOREM. *Let L be a graded restricted Lie algebra.*

(i) $\omega(L)$ is nilpotent if and only if L is finite dimensional.

(ii) $u(L)$ is strongly Lie nilpotent if and only if the restricted subalgebra of L generated by L' is finite dimensional.

We say that a Lie algebra L is $\mathcal{FC}$ if the centralizer $C_L(x)$ of each element $x \in L$ is of finite codimension in L. By a finitely generated restricted Lie algebra we mean a Lie algebra which is finitely generated as a restricted Lie algebra (but not necessarily as an ordinary Lie algebra).

4.1.5 THEOREM. *Let L be any graded restricted Lie algebra.*

(i) The following are equivalent:

(a) $u(L)$ is a PI–algebra,

(b) $u(L)$ is Lie n–Engel for some n, and

(c) L is Lie nilpotent and has a restricted ideal A such that L/A and A' are

finite dimensional and p–nilpotent.

(ii) Suppose also that L is finitely generated or $\mathcal{FC}$. Then $u(L)$ is PI–algebra if and only if $u(L)$ is strongly Lie nilpotent.

Clearly Theorem 4.1.5 in conjunction with Corollary 4.1.3 completely describes when $u(L)$ is Lie n–Engel for some n in the case when L admits a grading.

Finally, we apply our results to a class of graded enveloping algebras of particular interest to the study of group rings, one which we have in fact discussed in section 3.9. Let K be a field of characteristic $p > 0$, and G be any group. Consider the group ring KG, and let Δ be its augmentation ideal. Put

$$gr(KG) = \bigoplus_{m \geq 0} \Delta^m/\Delta^{m+1} \quad \text{and} \quad gr(KG)^+ = \bigoplus_{m \geq 1} \Delta^m/\Delta^{m+1},$$

viewed as graded associative K–algebras. Here $\Delta^0 = KG$ so that $\Delta^0/\Delta^1 \cong K$. We would like to determine exactly when $gr(KG)$, the graded group ring associated to KG, satisfies a polynomial identity. Specifically, when is $gr(KG)$ strongly Lie nilpotent or Lie n–Engel, for some n?

Group rings that are Lie nilpotent were characterized by Passi, Passman and Sehgal in [PPS]. The result in positive characteristic p shows that KG is Lie nilpotent (strongly or otherwise) if and only if G is nilpotent and the derived subgroup of G is a finite p–group. Subsequently in [S; V.6.1] Sehgal proved that KG is Lie n–Engel, for some n, if and only if G is nilpotent and possesses a normal subgroup A such that both G/A and A are finite p–groups. The analogy between the PI–theory of group rings and the PI–theory of restricted enveloping algebras is quite evident. Obviously, if KG satisfies a certain polynomial identity, then so does $gr(KG)$; however, as we shall see, the converse need not hold.

As mentioned section 3.9, in [Q], Quillen was able to present $gr(KG)$ as a restricted universal enveloping algebra $u(\mathcal{L}(G))$, where

$$\mathcal{L}(G) = K \otimes_{\mathbf{F}_p} \left(\bigoplus_{m \geq 1} D_m/D_{m+1} \right).$$

Here D_m is just an abbreviation for $D_m(G)$. Notice that then $gr(KG)^+$ is the augmentation ideal of $gr(KG)$.

Recently, Shalev [Sh3] has exploited this result of Quillen along with Passman's theorem to obtain the following very interesting result.

4.1.6 THEOREM. *The following are equivalent for any field K of positive characteristic p and any finitely generated group G:*

(i) $gr(KG)$ is a PI–algebra,

(ii) $\mathcal{L}(G)$ is nilpotent, and

(iii) the pro-p completion of G is p-adic analytic.

Shalev has pointed out that this result is strictly about finitely generated groups; however, we will give a simple description of the lower central series of $\mathcal{L}(G)$ for arbitrary groups G. Using this description and Theorems 4.1.4 and 4.1.5 we will obtain our last primary result. As usual, G_m denotes the mth term of lower central series of a group G.

4.1.7 THEOREM. *Let G be any group, and let K be any field of characteristic $p > 0$.*

(i) The following are equivalent:

(a) $gr(KG)$ is a PI–algebra,

(b) $gr(KG)$ is Lie n–Engel for some n, and

(c) $\mathcal{L}(G)$ is Lie nilpotent and has a restricted ideal A such that $\mathcal{L}(G)/A$ and A' are finite dimensional and p–nilpotent.

(ii) The following are equivalent:

(a) $gr(KG)^+$ *is nilpotent,*

(b) $\mathcal{L}(G)$ *is finite dimensional, and*

(c) $[G : D_\omega] < \infty$, *where* $D_\omega = \bigcap_{m \geq 1} D_m$.

(iii) The following are equivalent:

(a) $gr(KG)$ *is strongly Lie nilpotent,*

(b) $\mathcal{L}(G)'$ *is finite dimensional, and*

(c) $\sum_{m \geq 2} log_p[G_m D_{m+1} : D_{m+1}] < \infty$.

(iv) If G *is finitely generated, then the following coincide:*

(a) $gr(KG)$ *is a PI-algebra,*

(b) $gr(KG)$ *is strongly Lie nilpotent,*

(c) $\mathcal{L}(G)$ *is nilpotent, and*

(d) $G_m \subseteq D_{m+1}$ *for some* $m \geq 1$.

4.2 Lie nilpotent enveloping algebras.

In what follows, L represents a restricted Lie algebra. If S is a subset of L, then let S_p denote the restricted subalgebra generated by S. Let us write L'_p for $(L')_p$. We denote by $\mathcal{F}_p$ the class of finite dimensional p–nilpotent restricted Lie algebras over K. We also say an (ordinary) subalgebra S of L is $\mathcal{F}_p$ if S is finite dimensional and p–nilpotent. We will see immediately that this convention is not misguided. The following lemma is contained in [SF; 2.1.3].

4.2.1 LEMMA. *Let* S *be an (ordinary) subalgebra of* L.

(i) $S \in \mathcal{F}_p$ *if and only if* $S_p \in \mathcal{F}_p$.

(ii) If S is an ideal of L, then S_p is a restricted ideal.

If H is a restricted ideal of L, then it is well known that the kernel, $\omega(L,H)$, of the canonical map $u(L) \to u(L/H)$ is merely $\omega(H)u(L) = u(L)\omega(H)$. Let $\zeta(L)$ denote the centre of L.

4.2.2 LEMMA. *Suppose $L \in \mathcal{F}_p$. Then $\omega(L)$ is nilpotent.*

Proof. First let us prove that L is Lie nilpotent. Let $x \in L$. Since L is finite dimensional, it suffices by Engel's Theorem [SF; 1.3.4] to show that $ad\ x$ is nilpotent. But by assumption there is an $n \in \mathbb{N}$ such that $x^{[p]^n} = 0$, so that $(ad\ x)^{p^n} = ad\ x^{[p]^n} = 0$.

Now to prove $\omega(L)$ is nilpotent, we will proceed by induction on $dim_K L$. We may assume $L \neq 0$. Let $\zeta(L)$ denote the centre of L. Since L is Lie nilpotent, there exists a nonzero $z \in \zeta(L)$ with $z^{[p]} = 0$. The restricted Lie ideal of L generated by z, namely $\langle z \rangle_p$, is just Kz. Set $\overline{L} = L/\langle z \rangle_p$. Then $dim_K \overline{L} < dim_K L$. Consequently, by the induction hypothesis, there is an $m \in \mathbb{N}$ such that $\omega(\overline{L})^m = \overline{0}$. In other words, by the PBW theorem (see [SF; 2.5.1])

$$\omega(L)^m \subseteq \omega(L, \langle z \rangle_p) = \omega(\langle z \rangle_p)u(L) = (\bigoplus_{i=1}^{p-1} Kz^i)u(L) = zu(L).$$

Therefore, by the centrality of z, we have $\omega(L)^{mp} = 0$.

□

4.2.3 LEMMA. *Suppose that L is nilpotent and $L' \in \mathcal{F}_p$. Then $\omega(L)$ is strongly Lie nilpotent.*

Proof. This time we will use induction on $dim_K L'$. If $dim_K L' = 0$, then L, and hence $u(L)$, is abelian. Thus we may assume that $L' \neq 0$. Since L is nilpotent, we can then find a nonzero $z \in L' \cap \zeta(L)$ such that $z^{[p]} = 0$. Put $\overline{L} = L/\langle z \rangle_p$. Then

$dim_K \overline{L}' < dim_K L'$, so that by the induction hypothesis there exists an $m \in \mathbf{N}$ such that $\omega(L)^{(m)} \subseteq \omega(L, \langle z \rangle_p) = zu(L)$. A simple inductive argument using the centrality of z now yields that $\omega(L)^{(mp)} \subseteq z^p u(L) = 0$.

□

4.2.4 LEMMA. *(i)* $\omega((\gamma_m(L))_p) \subseteq \omega(L)^{(m)}$, *for all* $m \geq 1$;

(ii) $[\omega(L)^{(m)}, \omega(L)^{(n)}] \subseteq \omega(L)^{(m+n)}$, *for all* $m, n \geq 1$; *and*

(iii) $\omega(L)^{(m)}\omega(L)^{(n)} \subseteq \omega(L)^{(m+n-1)}$, *for all* $m, n \geq 1$.

Proof. For part (i), first use induction to show that $\gamma_m(L) \subseteq \omega(L)^{(m)}$. Then the rest follows. It can be shown that (ii) and (iii) hold for any associative ring R replacing $\omega(L)$. See [P; I.1.7] for a proof of the group ring version.

□

Finally, we will need the following pivotal result of Passman [P2; 5.1].

4.2.5 LEMMA. *If* L *is infinite dimensional, the right and left annihilators of* $\omega(L)$ *in* $u(L)$ *are zero.*

Proof of Theorem 4.1.2. (i) Suppose that $\omega(L)$ is nilpotent. Then there exists an m sufficiently large so that $\omega(L)^{p^m} = 0$. Hence $L^{[p]^m} = 0$ since $L \subseteq \omega(L)$, so that L is p-nilpotent. Suppose now, to the contrary, that L is infinite dimensional. We can find a least integer n with $\omega(L)^n = 0$, so that $\omega(L)^{n-1} \neq 0$. Pick $0 \neq x \in \omega(L)^{n-1}$. Then $x\omega(L) = 0$, so that x lies in the left annihilator of $\omega(L)$. But then by Lemma 4.2.5, x must be zero, a contradiction. Thus we have shown that $L \in \mathcal{F}_p$, and in deference to Lemma 4.2.2, this completes the proof of part (i).

(ii) Now suppose that $\omega(L)$ is strongly Lie nilpotent. Then there is some integer m with $\omega(L)^{(m)} = 0$. By Lemma 4.2.4(i) this implies that $\gamma_m(L) = 0$, so that L is Lie nilpotent. Next, by Lemma 4.2.4(i) and (iii), we have $\omega(L'_p)^{m-1} \subseteq (\omega(L)^{(2)})^{m-1} \subseteq$

$\omega(L)^{(m)} = 0$, so that by part (i), $L' \subseteq L'_p \in \mathcal{F}_p$. Finally, an appeal to Lemma 4.2.3 completes the proof of part (ii).

□

Proof of Corollary 4.1.3. First let $u(L)$ satisfy some polynomial identity. Then by Passman's theorem 4.1.1 and Theorem 4.1.2(i), we can find a restricted subalgebra A of L such that $dim_K L/A < \infty$, $\gamma_3(A) = 0$, and $A' \in \mathcal{F}_p$ since B contains A'. Sufficiency follows as well by Passman's theorem and Theorem 4.1.2(i) since Lemma 4.2.1(i) permits us to take B to be A'_p. Now we need only prove that we can choose A to be a graded if L admits a grading, but, as remarked before, this has already been done by Shalev [Sh3; 4].

□

4.3 Lie nilpotent graded enveloping algebras.

We will make use of the following result of Shalev [Sh3; 5.1].

4.3.1 LEMMA. *Suppose that L be a restricted Lie algebra, and let A be a restricted ideal of L such that $L/A \in \mathcal{F}_p$ and A is Lie nilpotent. Then L is Lie nilpotent.*

Recall that a Lie algebra R is Lie soluble if for some integer n, $\delta_n(R) = 0$, where, inductively, $\delta_0(R) = R$ and $\delta_{n+1}(R) = [\delta_n(R), \delta_n(R)]$.

4.3.2 LEMMA. *Let L be a restricted Lie algebra, such that $L' \in \mathcal{F}_p$. Then $u(L)$ is Lie soluble; in particular, $u(L)$ satisfies a polynomial identity.*

Proof. Consider the map $u(L) \to u(L/L'_p)$. The kernel of the map is $\omega(L, L'_p) = \omega(L'_p)u(L) = u(L)\omega(L'_p)$, which, by Theorem 4.1.2(i) and Lemma 4.2.1(i), is nilpotent, and hence Lie soluble. Thus, since the image of the map is commutative, this implies that $u(L)$ is Lie soluble.

□

4.3.3 LEMMA. *Suppose that L is a graded restricted Lie algebra.*

(i) If L is finite dimensional, then L is p–nilpotent.

(ii) If A is a graded restricted subalgebra L such that $dim_K L/A < \infty$, then A contains a restricted ideal I of L with $dim_K L/I < \infty$.

(iii) If $u(L)$ is a PI–algebra, then L is Lie nilpotent.

Proof. (i) Suppose that $L = \bigoplus_{m \geq 1} L_m$ is finite dimensional. Then for some $M \geq 1$, $L_m = 0$ whenever $m \geq M$. Thus, if $p^n \geq M$, we have $L_m^{[p]^n} \subseteq L_{mp^n} = 0$ for all m. Hence L is p–nilpotent.

(ii) Write $L = \bigoplus_{m \geq 1} L_m$ and $A = \bigoplus_{\geq 1} A_m$. Since $dim_K L/A < \infty$, then there exists some $M \geq 1$ such that $A_m = L_m$ whenever $m \geq M$. Hence $\gamma_M(L) \subseteq A$.

Think of L as an A–module. Then A is a submodule, so that L/A (as vector space quotient) is an A–module of finite dimension. Then $C_A(L/A) = \{a \in A | \, [a, L] \subseteq A\}$ is a restricted subalgebra of A of finite codimension in A and hence in L. Indeed, $A/C_A(L/A)$ has L/A as a faithful module. If $C_A(L/A) = A$, then A is an ideal of L. If not, inductively define for each $m \geq 1$, $I_1 = A$ and $I_m = C_{I_{m-1}}(L/I_{m-1})$. From the preceding argument, each I_m is a restricted subalgebra of A with finite codimension in L. But $I_M = \{a \in I_{M-1} | \, [a, {}_{M-1}L] \subseteq A\} = I_{M-1}$ since $\gamma_M(L) \subseteq A$. Hence, I_{M-1} is an ideal of L, so that we may take $I = I_{M-1}$.

(iii) By Corollary 4.1.3 and part (ii) above there exists a Lie nilpotent restricted ideal A of L with finite codimension in L. Since $dim_K L/A < \infty$, all the homogeneous components of L but, say, $L_{m_1}, \ldots, L_{m_t}$ are contained in A, where $m_1 < \cdots < m_t$. Choose an integer n such that $p^n m_1 > m_t$. Then $L^{[p]^n} \subseteq A$. Now applying Lemma 4.3.1, we see that L is Lie nilpotent.

□

Proof of Theorem 4.1.4. (i) Theorem 4.1.2(i) and Lemma 4.3.3(i) proves (i).

(ii) Suppose that L'_p is finite dimensional. Since L'_p is graded if L is, Lemma 4.3.3(i) shows that L'_p is p–nilpotent. To show that $u(L)$ is strongly Lie nilpotent, by Theorem 4.1.2(ii), it remains to show only that L is nilpotent. But by Lemma 4.3.2 we know that $u(L)$ is a PI–algebra, so that L is Lie nilpotent by Lemma 4.3.3(iii). □

4.4 Lie n–Engel graded enveloping algebras.

Let us call an element x of a restricted Lie algebra L $[p]$–independent if the nonzero elements of the set $\{x, x^{[p]}, x^{[p]^2}, \dots\}$ are linearly independent.

4.4.1 LEMMA. *The elements of a graded restricted Lie algebra L are $[p]$–independent.*

Proof. Let $x \in L$. Suppose, to the contrary, that there is an integer l, with

$$(*) \qquad x^{[p]^l} = \sum_{i>l} \alpha_i x^{[p]^i} \neq 0,$$

for some $\alpha_i \in K$ $(i > l)$. Then since L is graded, the least degree of the nonzero homogeneous components of the left side of $(*)$ is strictly less than the degree of every nonzero component on the right side of $(*)$. This is impossible.

□

Now consider the following lemma reminiscent of Schur's Lemma for groups.

4.4.2 LEMMA. *If L is an ordinary Lie algebra such that $dim_K L/\zeta(L) = m < \infty$ then $dim_K L' \leq m^2 - m$. Moreover, if L is restricted and $[p]$–independent, then $(L')^{[p]^r} = 0$, where $r \geq m^2 - m$.*

Proof. Let $\{x_1, x_2, \dots x_m\}$ be a basis for the complement of $\zeta(L)$ in L. Then any $x, y \in L$ can be written as $x = w + \sum_{i=1}^m \alpha_i x_i$ *and* $y = z + \sum_{j=1}^m \beta_j x_j$, where $\alpha_i, \beta_j \in K$ $(1 \leq i, j \leq m)$ and $w, z \in \zeta(L)$. Then $[x, y] = \sum_{i \neq j} \alpha_i \beta_j [x_i, x_j]$. Hence $dim_K L' \leq m^2 - m$ as claimed. The rest is obvious.

□

4.4.3 LEMMA. *If L is finitely generated as a restricted Lie algebra by d elements and A is a restricted ideal of L such that $dim_K L/A = n < \infty$, then A is finitely generated as a restricted ideal by $f(d,n) = d^2 + n^2 + dn + 2(d+n)$ elements.*

Proof. Suppose that $\{x_1, x_2, \ldots, x_d\}$ is a generating set for L and $\{\overline{y_1}, \overline{y_2}, \ldots, \overline{y_n}\}$ is a K-basis of $\overline{L} = L/A$. Put $M = \sum_{i=1}^{d} Kx_i + \sum_{j=1}^{n} Ky_j$, $M_1 = \gamma_2(M)$, and $M_2 = \sum_{i=1}^{d} Kx_i^{[p]} + \sum_{j=1}^{n} Ky_j^{[p]}$. Let $S = A \cap (M + M_1 + M_2)$. If $m_1, m_2 \in M$, then $[m_1, m_2] = m + a$, where $m \in M$ and $a \in S$. Indeed, $\overline{[m_1, m_2]} = \sum_{j=1}^{n} \alpha_j \overline{y_j}$, for some $\alpha_j \in K$ $(1 \leq j \leq n)$; so that if we take $m = \sum_{j=1}^{n} \alpha_j y_j \in M$, then $a = [m_1, m_2] - m \in S$. Let I be the restricted ideal of L generated by S. We just proved $\gamma_2(M) \subseteq M + I$, so that $\gamma_k(M) \subseteq M + I$ for all $k \geq 1$. We claim next that $M^{[p]} \subseteq M + I$. Indeed, for each i, $x_i^{[p]} = \sum_{j=1}^{n} \alpha_j y_j + a$ for some $\alpha_j \in K$ $(1 \leq j \leq n)$ and $a \in A \cap (M + M_2)$, so that $x_i^{[p]} \in M + I$. Similarly, each $y_j^{[p]} \in M + I$. Since we already know $\gamma_k(M) \subseteq M + I$, for all $k \geq 1$, the claim now follows.

Next let us prove that $M + I$ is a restricted ideal of L. Since $\gamma_2(M) \subseteq M + I$, it is clear that for each i, $[x_i, M + I] \subseteq M + I$. Thus, since the x_i are generators of L, it is easy to see that $M + I$ is an (ordinary) ideal of L. If $m \in M$ and $x \in I$, then $(m + x)^{[p]} \in M + I$ since $m^{[p]} \in M + I$ and $(m + x)^{[p]} \equiv m^{[p]} + x^{[p]} \equiv m^{[p]}$ modulo I. Hence, indeed, $M + I$ is a restricted ideal of L. But since $M + I$ contains the generators x_i of L, this must imply that $M + I = L$.

Now observe that $M + M_1 + M_2$ has dimension at most $l = d^2 + n^2 + dn + d + n$, so that S has dimension at most l. Thus I is finitely generated as a restricted ideal of L by l elements. Also notice that since $I \subseteq A$, we have $dim_K(A/I) \leq dim_K(L/I) \leq dim_K M \leq d + n$. Hence by adding the at most $d + n$ quotient space generators of A/I to the l generators of I, we see that A is generated by $l + d + n$ elements.

□

4.4.4 LEMMA. *If L is a graded Lie soluble restricted Lie algebra such that $L^{[p]^m} \subseteq \zeta(L)$, then $(L'_p)^{[p]^r} = 0$ for some fixed r.*

Proof. We use induction on the Lie solubility length of L. Put $L'' = \delta_2(L)$ and $L''_p = (L'')_p$. Observe that $L''_p = (L'_p)'_p$. Similarly, it is easy to see that L'_p and L' have the same Lie solubility class; hence, by the induction hypothesis, we have $L''^{[p]^l}_p = 0$ for a fixed l. Notice that L induces a grading on L''_p, so that L/L''_p is graded by [SF; p.117]. In particular then, L/L''_p is $[p]$–independent. Thus, we may factor by L''_p. Since L'_p is now abelian, it suffices to prove that for all $x, y \in L$, that $[x,y]^{[p]^M} = 0$ for a fixed M. Let $H = \langle x, y \rangle_p$. It is easy to see that $H^{[p]^m} \subseteq \zeta(H)$ forces $dim_K H/(H' + \zeta(H)) \leq 2m$. Also note $H'_p + \zeta(H) = (H' + \zeta(H))_p$. Since $H''_p = 0$, $H/(H'_p + \zeta(H))$ has $(H'_p + \zeta(H))/\zeta(H)$ as a restricted module with action defined by $(w + \zeta(H)) \cdot (z + H'_p + \zeta(H)) = [w,z] + \zeta(H)$ for all $w \in H'_p$ and $z \in H$. Hence, the orbit of $w + \zeta(H)$, namely $(w + \zeta(H)) \cdot u\left(H/(H'_p + \zeta(H))\right)$ has dimension at most p^{2m}. Indeed, $dim_K H/(H'_p + \zeta(H)) \leq 2m$ implies that $u\left(H/(H'_p + \zeta(H))\right)$ has dimension at most p^{2m} by the PBW theorem.

Now observe that Lemma 4.4.3 implies that $H'_p + \zeta(H)$ is generated as a restricted ideal of H by $n = f(2, 2m)$ elements. Collecting all the facts that we have deduced so far yields that $dim_K(H'_p + \zeta(H))/\zeta(H) \leq n(p^{2m})^2$. Therefore, $dim_K H/\zeta(H) \leq 2m + np^{4m}$. Now since H is $[p]$–independent, Lemma 4.4.2 implies that $(H')^{[p^r]} = 0$, where $r = (2m + np^{4m})^2 - 2m + np^{4m}$.

□

Recall the following well known fact, which is actually one of the defining properties of a restricted Lie algebra. See [S; V.4.3] for a proof of the associative case.

4.4.5 LEMMA. *In any ring R of prime characteristic p, we have $[x, {}_{p^m}y] = [x, y^{p^m}]$, for all $x, y \in R$. In particular, R is Lie p^m–Engel if and only if $x^{p^m} \in \zeta(R)$ for each $x \in R$.*

Proof of Theorem 4.1.5(i). Suppose that $u(L)$ satisfies some polynomial identity. Then by Corollary 4.1.3 and Lemma 4.3.3, L is Lie nilpotent with a restricted ideal A such that L/A and A' lie in $\mathcal{F}_p$. We only use these facts about L in what follows. Let $dim_K L/A = n$. Since L is Lie nilpotent, it is certainly Lie p^m–Engel for some m. Thus $L^{[p]^m} \subseteq \zeta(L)$.

We will proceed by induction on $dim_K A'_p$, which is finite by Lemma 4.2.1(i). If $dim_K A'_p > 0$, then, since L is Lie nilpotent, we can find an element $0 \neq z \in A'_p \cap \zeta(L)$ such that $z^{[p]} = 0$. Let $\overline{L} = L/\langle z \rangle_p$. Then $\overline{A}'_p = A'_p/\langle z \rangle_p$ has dimension less than A'_p. Hence, by induction, for all $x, y \in u(L)$ there exists a fixed μ such that $[x, y^{p^\mu}] \in \omega(L, \langle z \rangle_p) = zu(L)$. It is now an easy matter to see $[x, y^{p^{\mu+1}}] = [x, {}_p y^{p^\mu}] \in z^p u(L) = 0$. Thus, we may assume that A is abelian.

Let $w \in \omega(L)$. If $\{x_j \mid j \in J\}$ is an ordered basis of L, then by the PBW theorem

$$w = \sum \alpha x_{j_1}^{s_1} x_{j_2}^{s_2} \cdots x_{j_t}^{s_t},$$

where $\alpha \in K$, $t > 0$, $j_1 < j_2 < \cdots j_t$ in J, and $0 \leq s_k \leq p-1$ $(0 \leq k \leq t)$. Hence

$$w^{p^m} \equiv \sum \alpha^{p^m} (x_{j_1}^{p^m})^{s_1} (x_{j_2}^{p^m})^{s_2} \cdots (x_{j_n}^{p^m})^{s_t} \text{ modulo } [\omega(L), \omega(L)],$$

so that $w^{p^m} = \theta + \eta$, where $\theta \in \zeta(u(L))$ and $\eta \in \omega(L, L'_p)$.

Next observe that Lemma 4.4.4 implies that $(L'_p)^{[p]^r} = 0$ for a fixed r. Set $B = [L, A]_p$. Then, in particular, L'/B is p–nilpotent. Also, L'/B is finite dimensional by Lemma 4.4.2 since $A/B \subseteq \zeta(L/B)$ and $\frac{L/B}{A/B} \cong L/A$ is finite dimensional. Thus, by Lemma 4.2.1(i), $L'_p/B = (L'/B)_p \in \mathcal{F}_p$. Now by Theorem 4.1.4(i), we have that $\omega(L'_p/B)$ is nilpotent, so that there exists an integer l such that $\omega(L'_p)^{p^l} \subseteq \omega(L'_p, B) \subseteq \omega(L, B)$. Consequently,

$$(w^{p^m})^{p^l} = \theta^{p^l} + \eta^{p^l} + \beta$$

for some $\beta \in \omega(L,B)$.

Clearly $B^{[p]^r} = 0$ since $B \subseteq L_p'$. Then since $\omega(A,B)$ is commutative, $\alpha^{p^r} = 0$ for all $\alpha \in \omega(A,B)$. We now claim that if $\beta \in \omega(L,B)$, then $\beta^{p^M} \in \zeta(u(L))$, where $p^M \geq p^r(p^n)^{p^n}$. Let $\{x_1, x_2, \ldots, x_n\}$ be an (ordered) basis for the complement of A in L. Then it is not difficult to see that

$$u(L) = \sum_{0 \leq s_1, s_2, \ldots s_n \leq p-1} u(A) x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}.$$

Hence we can write $\beta = \sum \beta_{(s_1,s_2,\ldots,s_n)} x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}$, where each $\beta_{(s_1,s_2,\ldots,s_n)} \in \omega(A,B)$. Using PBW and the identity $[ab,c] = a[b,c] + [a,c]b$, it is easy to check that $[\omega(A), L] \subseteq \omega(A)$. Thus $\overline{L} = L/A$ has $\omega(A)$ as a restricted module under the action $\alpha \cdot \overline{x} = [\alpha, x]$, for all $x \in L$ and $\alpha \in \omega(A)$. Then the orbit of α, namely $\alpha \cdot u(\overline{L})$, has dimension at most p^n since $u(\overline{L})$ has dimension p^n by PBW.

Let R_β be the ring generated by all the $\beta_{(s_1,s_2,\ldots,s_n)} \cdot u(\overline{L})$. Then R_β is a commutative ring contained $\omega(A,B)$ generated by $(p^n)^{p^n}$ elements. Therefore $R_\beta^{p^M} = 0$ where $p^M \geq p^r(p^n)^{p^n}$ (see [S; V.6.4] for an easy proof). That $\beta^{p^M} = 0$ now follows by repeated use of the identity $xb = bx - [b,x]$ and noticing that $[\beta_{(s_1,s_2,\ldots,s_n)}, x_{i_1}, x_{i_2}, \ldots x_{i_t}] \in R_\beta$ for all $\beta_{(s_1,s_2,\ldots,s_n)}$, $i_1, i_2, \ldots i_t \in \{1,2,\ldots n\}$, and $t > 0$.

Now, putting everything we have deduced together, we see

$$w^{p^{m+l+M}} = \theta^{p^{l+M}} + \eta^{p^{l+M}} + \beta^{p^M} \in \zeta(u(L))$$

since

$$\beta^{p^M} = 0 \text{ and } \eta^{p^{l+M}} = (\eta^{p^l})^{p^M} \in (\omega(L_p')^{p^l} u(L))^{p^M} \subseteq \omega(B)^{p^M} u(L) = 0.$$

Hence, by Lemma 4.4.5, $u(L)$ is Lie p^{m+l+M}–Engel since m, l and M do not depend on w.

□

Let us now state the following simple result. See [P2; 6.2] for a proof.

4.4.6 LEMMA. *If a restricted Lie algebra L is $\mathcal{FC}$ and $dim_K L/A < \infty$, then $dim_K A' < \infty$ if and only if $dim_K L' < \infty$.*

We will need another result of Shalev [Sh3; 5.2].

4.4.7 LEMMA. *The following are equivalent for a graded finitely generated restricted Lie algebra L:*

(i) L has Lie nilpotent restricted subalgebra of finite codimension;

(ii) L is Lie nilpotent; and

(iii) $L/\zeta(L)$ is finite dimensional.

Proof of Theorem 4.1.5(ii). One direction is trivial. So suppose $u(L)$ is a PI–algebra. If L is $\mathcal{FC}$, then, by Theorem 4.1.5(i) and Lemma 4.4.6, we have that L is Lie nilpotent and $dim_K L' < \infty$. Thus, by Lemmas 4.4.1 and 4.4.4, $L' \in \mathcal{F}_p$. Now using Theorem 4.1.4(ii) and Lemma 4.2.1(i) we see that $u(L)$ is strongly Lie nilpotent.

On the other hand, suppose that L is finitely generated. Since $u(L)$ is a PI–algebra, by Theorem 4.1.5(i), L is Lie nilpotent. Hence, by Lemma 4.4.7, $L/\zeta(L)$ is finite dimensional, so that L is $\mathcal{FC}$. This reduces us to the case already proven.

□

4.5 Graded group rings.

We will need the following lemma, which is comprised of results of Lazard [L] and Passi and Sehgal [PS] already mentioned in Chapters Two and Three. In what follows, G and K are fixed and are dropped from much of the notation. As usual, round brackets (,) denote group commutators.

4.5.1 LEMMA. *For all $j, m, n \geq 1$ we have:*

(i) $D_m = \prod_{ip^j \geq m} G_i^{p^j}$, $(D_m, D_m) \subseteq D_{m+n}$, and $D_m^{p^j} \subseteq D_{mp^j}$; *and*

(ii) $D_{(m)} = \prod_{(i-1)p^j \geq m-1} G_i^{p^j}$, $(D_{(m)}, D_{(n)}) \subseteq D_{(m+n)}$, and $D_{(m)}^{p^j} \subseteq D_{((m-1)p^j+1)}$.

The following proposition is a special case of Theorem 3.3.9. We offer a new proof that is independent of the results of Chapter Three.

4.5.2 PROPOSITION. $(D_m, D_n) \subseteq D_{(m+n)}$ *for all* $m, n \geq 1$.

Proof. The group commutator identity $(xy, z) = (y, z)^x(x, z)$ and Lemma 4.5.1(i) enable us to reduce the problem to proving that $(x^{p^j}, y) \in D_{(m+n)}$ for all $y \in D_n$ and $x \in G_i$ such that $ip^j \geq m$. Let $z = (x, y)$. Then by Lemma 4.5.1(ii), $z \in D_{(i+n)}$. Notice that $(x^{p^j}, y) = x^{-p^j}(xz)^{p^j}$. Using the collection process of Hall [H; 12.3.1], we can collect all the basic commutators in x and z of weight less than $m+n$ to obtain:

$$(xz)^{p^j} = x^{p^j} z^{p^j} c_1^{e_1} c_2^{e_2} \cdots c_s^{e_s} R,$$

where R lies in $G_{m+n} \subseteq D_{(m+n)}$ and $c_1, c_2, \ldots c_s$ are basic commutators of weights $w_1, w_2, \ldots w_s$, respectively, in x and z. Thus, since each c_l has at least one occurrence of z, by Lemma 4.5.1(ii), $c_l \in D_{(i+n+(w_l-1)i)} = D_{(n+w_l i)}$. The exponents e_l $(1 \leq l \leq s)$ are of the form $b_1\binom{p^j}{1} + \cdots b_{w_l}\binom{p^j}{w_l}$, where the $b's$ are nonnegative integers. Let $|r|_p$ denote the p–part of an integer r. Choose an integer r_l, $1 \leq r_l \leq w_l$, such that $\left|\binom{p^j}{r_l}\right|_p$ is minimal. Then it is well known [P1; p.482] that $\frac{p^j}{|r_l|_p} \left| \left|\binom{p^j}{r_l}\right|_p \right.$, so that $\frac{p^j}{|r_l|_p} \Big| e_l$. Now, by Lemma 4.5.1(ii), we have that for each l,

$$c_l^{e_l} \in D_{(n+w_l i)}^{\frac{p^j}{|r_l|_p}} \subseteq D_{(m+n)}.$$

Indeed, $(n + w_l i - 1)\frac{p^j}{|r_l|_p} + 1 = \frac{w_l}{|r_l|_p} ip^j + (n-1)\frac{p^j}{|r_l|_p} + 1 \geq m + n$. Therefore, we have shown that $(x^{p^j}, y) \equiv z^{p^j}$ modulo $D_{(m+n)}$. However, $z^{p^j} \in D_{(i+n)}^{p^j} \subseteq D_{(m+n)}$, since $(i + n - 1)p^j + 1 \geq m + n$.

□

4.5.3 LEMMA. *$D_{(m)}$ is congruent to G_m modulo D_{m+1}.*

Proof. By Lemma 4.5.1(ii), $D_{(m)} = \prod_{(i-1)p^j \geq m-1} G_i^{p^j}$. So, suppose i and j satisfy $(i-1)p^j \geq m-1$. If $i < m$, then $j > 0$. Hence, by Lemma 4.5.1(i), $G_i^{p^j} \subseteq D_{ip^j} \subseteq D_{m+1}$, since $ip^j \geq m + (p^j - 1) \geq m + 1$.

□

4.5.4 PROPOSITION. *For all $k \geq 1$,*

$$\gamma_{k+1}(\mathcal{L}(G)) = K \otimes_{\mathbf{F}_p} \left(\bigoplus_{m \geq k+1} \frac{G_m D_{m+1}}{D_{m+1}} \right).$$

Proof. Let $E_{m,k} = D_m$ if $k = 0$ and $E_{m,k} = G_m$ if $k \geq 1$. We show by induction on $k \geq 0$ that

$$\gamma_{k+1}(\mathcal{L}(G)) = K \otimes_{\mathbf{F}_p} \left(\bigoplus_{m \geq k+1} \frac{E_{m,k} D_{m+1}}{D_{m+1}} \right).$$

The basic step is trivial by design. Suppose then that result holds for $k+1$. It is clear that homogeneous components of degree $t < k+2$ in $\gamma_{k+1}(\mathcal{L}(G))$, namely $(\gamma_{k+1}(\mathcal{L}(G)))_t$, are all zero. If $t \geq k+2$, then by the induction hypothesis

$$(\gamma_{k+1}(\mathcal{L}(G)))_t = \frac{(\prod_{n+m=t}(D_n, E_{m,k} D_{m+1}))D_{t+1}}{D_{t+1}}.$$

But by Proposition 4.5.2, if $n + m = t$, then $(D_n, E_{m,k} D_{m+1}) \subseteq (D_n, D_m) \subseteq D_{(m+n)} = D_{(t)}$. Therefore by Lemma 4.5.3,

$$(*) \qquad (\gamma_{k+2}(\mathcal{L}(G)))_t \subseteq \frac{D_{(t)} D_{t+1}}{D_{t+1}} = \frac{G_t D_{t+1}}{D_{t+1}}.$$

On the other hand, $G_t \subseteq (D_1, E_{t-1,k})$, so that equality holds in $(*)$. This proves the proposition.

□

4.5.5 COROLLARY. *$\gamma_{k+1}(\mathcal{L}(G))$ has dimension $\sum_{m\geq k+1} log_p[G_m D_{m+1} : D_{m+1}]$ over K, for each $k \geq 1$.*

We are now in a position to state precisely when $\mathcal{L}(G)$ is Lie nilpotent.

4.5.6 THEOREM. *The following are equivalent:*

(i) $\mathcal{L}(G)$ has Lie nilpotence class m,

(ii) $G_m \subseteq D_{m+1}$,

(iii) $\Delta^{(m)} \subseteq \Delta^{m+1}$, and

(iv) $\gamma_m(\triangle) \subseteq \Delta^{m+1}$.

Proof. (i)⇒(ii) follows by Proposition 4.5.4. (ii)⇒(i) follows by Lemma 4.5.1(i) since if $G_m \subseteq D_{m+1}$, then $G_{m+t} \subseteq D_{m+t+1}$ for all $t \geq 1$. (ii)⇒(iii) is an immediate consequence of Lemma 3.1.1. (iii)⇒(iv) is trivial. (iv)⇒(ii) follows by Lemma 2.2.1.

Proof of Theorem 4.1.7. (i) is a consequence of Theorem 4.1.5. (ii) follows from Theorem 4.1.4(i) since $dim_K \mathcal{L}(G) = log_p[G : D_\omega]$. For (iii), suppose that $\mathcal{L}(G)'$ finite dimensional. Then $\mathcal{L}(G)$ is Lie nilpotent by Corollary 4.5.5 and Theorem 4.5.6. Now it is easy to see using the grading of $\mathcal{L}(G)$ and Proposition 4.5.4 that $\mathcal{L}(G)'$ is p–nilpotent. Thus, by Lemma 4.2.1(i), the restricted subalgebra generated by $\mathcal{L}(G)'$ lies in $\mathcal{F}_p$. The rest follows from Theorem 4.1.4(ii) and Corollary 4.5.5. (iv) can be derived from Theorems 4.1.5(ii), 4.1.6, and 4.5.6.

□

REFERENCES

[BK1] A.A. Bovdi and I.I. Khripta, Generalized Lie nilpotent group rings, *Math. USSR Sbornik* **57** (1987), no. 1, 165–169.

[BK2] A.A. Bovdi and I.I. Khripta, Generalized nilpotency of the multiplicative group of a group ring, *Ukrainian Mathematical Journal* **41** (1990), no. 9, 1016–1020.

[F] T. Furukawa, A note on isomorphism invariants of a modular group algebra, *Math. J. Okayama Univ.* **23** (1981), 1–5.

[G1] N.D. Gupta, *Free Group Rings* (Contemporary Mathematics, **66**, AMS, 1987).

[G2] N.D. Gupta, The dimension subgroup conjecture, to appear in *Bull. AMS*.

[GL] N.D. Gupta and F. Levin, On the Lie ideals of a ring, *J. Algebra* **81** (1983), 225–231.

[GS] N.D. Gupta and J.B. Srivastava, Some remarks on the Lie dimension subgroup conjecture, preprint.

[H] M. Hall, *The Theory of Groups* (MacMillan, New York, 1959).

[Ha] B. Hartley, The residual nilpotence of wreath products, *Proc. London Math. Soc.* **2** (1970), 365–392.

[He] I.N. Herstein, *Noncommutative group rings* (MAA, Menasha, 1968).

[HS] T.C. Hurley and S.K. Sehgal, On the Lie diension subgroup conjecture, preprint.

[J] S.A. Jennings, On rings whose associated Lie rings are nilpotent, *Bull. AMS* **53** (1947), 593–597.

[K] I.I. Khripta, Nilpotency of the multiplicative group of a group ring, Thesis, Uzgorod (1971).

[L] M. Lazard, Sur les groupes nilpotents et les anneaux de Lie, *Ann. Ecole Norm. Sup.* **71** (1954), 101–190.

[LS] F. Levin and S.K. Sehgal, On Lie nilpotent group rings, *J. Pure Appl. Algebra* **37** (1985), 33–49.

[Li] A.I. Lichtman, The residual nilpotence of the augmentation ideal and the residual

nilpotence of some classes of groups, *Israel J. Math.* **26** (1977), 276–293.

[MW] I. Musson and A. Weiss, Integral group rings with residually nilpotent unit groups, *Arch. Math.* **38** (1982), 514–530.

[PrPS] M.M. Parmenter, I.B.S. Passi, and S.K. Sehgal, Polynomial ideals in group rings, *Canad. J. Math.* **25** (1973), 1174–1182.

[P] I.B.S. Passi, *Group Rings and Their Augmentation Ideals* (Lecture notes in math., **715**, Springer–Verlag, Berlin-Heidelberg-New York, 1979).

[PPS] I.B.S. Passi, D.S. Passman, and S.K. Sehgal, Lie solvable group rings, *Canad. J. Math.* **25** (1973), 748–757.

[PS1] I.B.S. Passi and S.K. Sehgal, Isomorphism of modular group algebras, *Math. Z.* **129** (1972), 65–73.

[PS2] I.B.S. Passi and S.K. Sehgal, Lie dimension subgroups, *Comm. Algebra* **3** (1975), 59–73.

[P1] D.S. Passman, *The algebraic structure of group rings* (Robert E. Krieger Pub., Malabar, 1985).

[P2] D.S. Passman, Enveloping algebras satisfying a polynomial identity, *J. Algebra* **134** (1990), 469–490.

[Q] D.G. Quillen, On the associated graded ring of a group ring, *J. Algebra* **10** (1968), 411–418.

[R] E. Rips, On the fourth integer dimension problem, *Israel J. Math.* **12** (1972), 342–346.

[RS] J. Ritter and S.K Sehgal, Isomorphism of group rings, *Arch. Math* **40** (1983), 32–39.

[R1] D.J.S. Robinson, *Finiteness conditions and generalized soluble groups* (Ergebnisse der Math. vol. 62, 63, Springer–Verlag, New York, 1972).

[R2] D.J.S. Robinson, *A course in the theory of groups* (Springer–Verlag, New York, 1980).

[Sa1] R. Sandling, The dimension subgroup problem, *J. Algebra* **21** (1972), 216–231.

[Sa2] R. Sandling, The modular group algebra of a central-elementary-by-abelian *p*-

group, *Arch. Math.* **52** (1989), 22–27.

[S] S.K. Sehgal, *Topics in group rings* (Marcel Dekker, New York, 1978).

[Sh1] A. Shalev, Dimension subgroups, nilpotency indices, and the number of generators of ideals in p-group algebras, *J. Algebra* **129** (1990), 412–438.

[Sh2] A. Shalev, Lie dimension subgroups, Lie-nilpotency indices, and the exponent of the group of normalized units, to appear in *J. London Math. Soc.*

[Sh3] A. Shalev, Polynomial identities in graded group rings, restricted Lie algebras and p-adic analytic groups, to appear in *Trans. AMS.*

[SS] R.K. Sharma and J.B. Srivastava, Lie ideals in group rings, *J. Pure Appl. Algebra* **63** (1990), 67–80.

[SF] H. Strade and R. Farnsteiner, *Modular Lie Algebras and Their Representations* (Marcel Dekker, New York, 1988).

[T] M.J. Tomkinson, *FC-groups* (Pitman Advanced Pub. Program, Boston, 1984).

[W] H.N. Ward, Some results on the group algebra of a group over a prime field, *Seminar in Finite Groups and Related Topics*, Mimeographed notes, Harvard Univ. (1960-61), 13–19.